SOLUTION OF SCALER RIEMANN HILBERT PROBLEMS

Amna Eltahir Mokhtar Eltahir[1], Prof. Dr. Abdel Radi Abdel Rahman Abdel Gadir Abdel Rahman[2*]

[1,2]*Department of Mathematics, Faculty of Education, Omdurman Islamic University, Omdurman, Sudan.*

Corresponding Author :
Email:dibdelradi78@oiu.edu.sd

Abstract:
We gave a mathematical formulation of the Riemann Hilbert problem . Our aimed was proceed to show scaler and the solution of scaler of Riemann-Hilbert problem arises in the theory. We followed the analytical mathematical method to solve some problems and we found that the solution of Scaler Riemann Hilbert problem depend to Holder condition, pemelj formula and Hardy spaces.

Key words: *Complex Plan, Holder Condition, Riemann Hilbert, Scaler, Smooth Closed, Smooth arc, Plemelj Formula, Hardy Spaces.*

1.Introduction:

In Mathematic, Riemann–Hilbert problems, named after a Riemann and Hilbert are a class of problems that arise in the study of differential equation in the complex plan. Several existence Theorem for Riemann–Hilbert problems have Holder condition produced. the problem is closely related to Riemann's idea that any function is completely determined by specifying its singularities and behave our round these singularities, it got the name Riemann-Hilbert problem. the Riemann-Hilbert method was developed. For an overview of these development the solution is given by exact formulae, its structure is not elementary.[10], p1

2.The Cauchy Integral Formula:

Here we develop the general version of the Cauchy integral formula valid for arbitrary closed rectifiable curves. The key idea in this development is the notion of the winding number. This is the number defined in the following theorem, also called the index. We make use of this winding number along with the earlier results, especially Liouville's theorem, to give an extremely general Cauchy integral formula.

Theorem (2.1): Let $\gamma : [a, b] \rightarrow C$ be continuous and have bounded variation with $\gamma(a) = \gamma(b)$. Also suppose that $z \notin \gamma([a, b])$. We define

$$n(\gamma, z) \equiv \frac{1}{2\pi i}\int_{\gamma} \frac{w}{w-z}\, dw.$$

Then $n(\gamma, z)$ is continuous and integer valued. Furthermore, there exists a sequence, $\eta k : [a, b] \rightarrow C$ such that ηk is $C([a, b])$. [9]p426

3.The Riemann-Hilbert problem:

Riemann-Hilbert problems (RHP), named after the giant German mathematicians Bernhard Riemann and David Hilbert, are a class of models which can be used to solve certain differential equations with the assistance of complex analysis techniques such as analytical continuation. RHP can be presented in slightly different ways when treating different problem. Here we introduce only the most typical representation that is widely used in integrable systems.[2], p30

Let $\gamma \subset C$ be an oriented contour in the complex λ-plane. The orientation defines traditionally the D^+, and D^-, sides of γ as being on the left and right sides of the direction arrow, respectively. Let G be a map from γ into the set of $N \times N$ invertible matrices which we shall denote by $GLN(C)$.

An RHP associated with the pair $(\gamma; G)$ consists in finding an N × N matrix-valued function $\Phi(\lambda)\ (\lambda \in C)$ characterized by

- $\Phi(\lambda)$ is holomorphic in $C\backslash\gamma$;
- $\Phi^+(\lambda) = G(\lambda)\Phi^-(\lambda)$ for all $\lambda \in \gamma$, where

$$\Phi^+(\lambda) = \lim_{\lambda' \to \lambda, \lambda' \in D_+} \Phi(\lambda'), \text{ and } \Phi^-(\lambda) = \lim_{\lambda' \to \lambda, \lambda' \in D_-} \Phi(\lambda'), \quad (1)$$

$G(\lambda)$ involved here is often called the jump matrix in this model; more generally, $\Phi^+(\lambda)$ and $\Phi^-(\lambda)$ can be defined as $\Phi(\lambda)$ restricted to $\lambda \in D^+$, and $\lambda \in D^-$, respectively (hence $\Phi^+(\lambda)$and $\Phi^-(\lambda)$ are holomorphic in D^+ and D^-, respectively;

- Both $\Phi^+(\lambda)$ and $\Phi^-(\lambda)$ approach the identity matrix as $\lambda \rightarrow \infty$ (canonical normalization condition.[2], p31

To solve the simplest scalar case $N = 1$, one can rewrite the original multiplicative jump condition into an additive form with the help of the logarithmic function

$$\log\Phi_+(\lambda) = \log\Phi_-(\lambda) + \log G(\lambda), \quad (2)$$

which can always be solved by using the Cauchy-Plemelj-Sokhotskii formula

$$\log \Phi(\lambda) = \frac{1}{2\pi i}\int \frac{\log G(z)}{z-\lambda}\, dz. \quad (3)$$

RHPs with $N \geq 1$ can be solved also explicitly by (3), whenever the involved matrix multiplication for G is abelian, i.e when $[G(\lambda_1), G(\lambda_2)] = G(\lambda_1)G(\lambda_2) - G(\lambda_2)G(\lambda_1) = 0$ for all $\lambda_1, \lambda_2 \in \gamma$; that is

$$\Phi(\lambda) = \exp \Phi(\lambda) = \frac{1}{2\pi i}\int\int_L^* \frac{\log G(z)}{z-\lambda}\, dz. \quad (4)$$

For a more general non-abelian matrix RHP, formula (3) or (4) unfortunately ceases to work–it is so far believed that in such cases the RHP cannot be solved in analytical form by means of contour integrals.

the first part of this proposition involving uniform convergence, we obtain [13], p31

$$f(z) = \frac{1}{2\pi i}\int_{\gamma}^* \frac{f(w)}{w-z}\, dw. \quad (5)$$

Definition (3.1): Suppose that we are given a simple smooth closed contour L dividing the plane of the complex variable into an interior domain and an exterior domain D^-, and two functions of on the contour, $G(t)$ and $g(t)$ which satisfy the Holder condition, where G(t) does not vanish. It is required to find two functions: $\Phi^+(z)$, analytic in the domain D^+, and $\Phi^-(z)$, analytic in the domain D^-, including $z = \infty$, which satisfy on the contour L either the linear relation

$$\Phi^+(z)\ G(t)\Phi^-(z) \quad (6)$$

or

$$\Phi^+(z)\ G(t)\Phi^-(z) + g(t) \quad (7)$$

The function ***G(t)*** will be called the coefficient of the Riemann problem, and
the function ***g(t)*** its free term Holder theory of Cauchy integrals the fundamental object of study in the theory of RH problems is the Cauchy integral.[2] p30
The Cauchy integral maps functions on a contour to analytic functions off the contour. We shall see later that under specific regularity conditions these functions can be put into a one-to-one correspondence. In this way, Cauchy integrals are critical in the solution of RH problems from both a numerical and an analytical perspective.[1], p114
As in the precise statement of an RH problem, we must understand the limiting values of (6), specifically issues related to existence and regularity. We describe a class of functions for which the Cauchy integral has nice properties

Definition (3.2): Let $a \in L^{\infty}$ and $1 < p < \infty$. The Riemann-Hilbert problem (RHP) in Hardy spaces is the problem of finding $\varphi, \psi \in H^{p}$ *(D)* for which

$$\varphi^{*} = a\overline{\varphi^{*}} \qquad a \text{ on } T \qquad (8)$$

where φ^{*} denotes the nontangential boundary values *of*φ . The following well-known result essentially shows that the study of Toeplitz operators is closely related to the RHP in Hardy spaces.[13], p31
We give the proof for completeness because it is not readily available in the literature. Let us first recall. a couple of useful results. For $f \in L^{1}$, define

$$f(z) = \frac{1}{2\pi i}\int_{L}^{*} \frac{f(\tau)}{\tau - z}\, d\tau,\ z \in c \setminus T \qquad (9)$$

Proposition (3.3): Let $a \in L^{\infty}$ and $1 < p < \infty$. Then the Riemann-Hilbert problem There fore the RHP (4)is equivalent to the following

$$\varphi^{*} = a\overline{\varphi^{*}}$$

and the problem of finding f in $\ker T_{a}$ are equivalent in H^{p}whilst also being bounded at each of the endpoints. Hence this is the sought solution of the homogeneous Riemann-Hilbert problem, and we note that it is correct up to multiplication by an arbitrary polynomial. Once again though, whatever [4], p3polynomial we choose, we have

$$\Phi^{+}(t)Y(t)\Phi^{-}(z) \qquad (10)$$

$$Y^{+}(t)\ G(t)\Phi^{-}(z) + g(t) \qquad (11)$$

where

$$Y^{\pm}(t) = \Pi(t)\ \exp[\pm \log G(t) \frac{1}{2\pi i}\int_{L}^{*} \frac{G(\tau)}{\tau - T}\, d\tau] \qquad (12)$$

Now the function $\varphi(\tau) = g(\tau)$,$Y^{\pm}(\tau)$ satisfies a Holder condition everywhere on L, except at the endpoints a_j $and b_j$, where it has integrable singularities of the
form respectively. Hence by the Plemelj formula, the general solution to the inhomogeneous problem is:

$$\Phi(z) = Y(z)\int \frac{1}{2\pi i}\int_{L}^{*} g(\tau)\, d\tau \qquad (13)$$

Theorem (Plemelj) (3.4): Let L be a simple smooth arc which, if closed, is traversed in the counter-clockwise direction. If $\varphi(\tau)$ is a function satisfying a Holder condition on L, then

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\varphi(t)\ \frac{1}{2\pi i}\int_{L}^{*} \frac{\varphi(T)}{T - t}\, dT \qquad (14)$$

or, equivalently,

$$\Phi^{+}(t)\Phi^{-}(t) = \varphi(t), \qquad (15)$$

$$\Phi^{+}(t)\Phi^{-}(t) = \frac{1}{2\pi i}\int_{L}^{*} \frac{\varphi(T)}{T - t}\, dT \qquad (16)$$

However, reformulating the original problem into an RHP still makes much sense, since it can always be reduced to the study of a linear singular-integral equation. Indeed, nonabelian RHPs usually arise when the original problem is nonlinear, so the value of the Riemann-Hilbert reformulation lies in the fact that it linearizes a nonlinear system effectively.
The Riemann-Hilbert approach has acquired wide applications in integrable systems, orthogonal polynomials, random matrices, and asymptotic analysis. In particular for many integrable systems, the inverse spectral or inverse scattering problems associated particularly with the Cauchy problems for *1+1* dimensional PDEs, or the construction of soliton solutions for these systems, can be formulated as RHPs on the real line R, [6] p5
The already obtained solutions to the Riemann-Hilbert problem can be used in order to solve linear equations with involutions in Leibniz algebras with logarithms. Namely,[3], p_{642}

4.Scalar Riemann-Hilbert problems:

A Riemann-Hilbert (RH) problem is a jump problem for a piecewise analytic function. The setting is the following, let γ be an oriented smooth contour in the complex plane. The orientation induces ***a+*** side and ***a−***side on **γ,** where the +side lie to the left and the −side to the right if one traverses the contour according to the orientation. The contour may have end-points or points of self-intersection. At such points the + and −sides are not defined. We use$\gamma_0$to denote the contour γ without the end-points and the points of self-intersection. in what follows, we shall be interested in Cauchy integrals of the form

$$\Phi(z) = \frac{1}{2\pi i}\int_{L}^{*} \frac{\varphi(\tau)}{\tau - z}\, d\tau, \qquad (17)$$

where L is assumed to be a simple smooth arc and $\varphi(\tau)$ is a function satisfying a Holder condition on L. the scaler Riemann Hilbert problem is the function theoretical problems or finding single function Φ which is sectionally analytic in $C_{\pm}$bounded and its corresponding upper and lower radial say$\Phi_{\pm}$having a prescribed jump dis continuity on real line $\Phi_{\pm}(w) = G$ it is dial limits of sokhotski plemeij integral of tow functions.[5], p5

Definition (4.1): A smooth arc is a differentiable function $\gamma(t) : t \in [\alpha, \beta] \to C$,where $\gamma'(t)$ is continuous and non-zero for $t \in (\alpha, \beta)$, continuous from the right at $t = \alpha$, and continuous from the left at $t = \beta$. A simple arc is one which does not intersect itself, except in the case $\gamma(\alpha) = \gamma(\beta)$, when the arc is closed.

Definition (4.2): The principal value of the integral $\Phi(t) = \frac{1}{2\pi i}\int_L^* \frac{\varphi(T)}{T-t}\, dT$ is defined to be

$$\lim_{\varepsilon\to 0}\int_{L\to L_\varepsilon}^* \frac{\varphi(T)}{T-t}\, dT \equiv \int_L^* \frac{\varphi(T)}{T-t}\, dT \qquad (18)$$

where L_ε is that part of L inside a circle of radius ε centred on t. Remark More generally, $\Phi(t)$ is said to exist in the Riemann sense if

$$\lim_{\varepsilon,\eta\to 0}\int_{L\to L_{\varepsilon,\eta}}^* \frac{\varphi(T)}{T-t}\, dT \qquad (19)$$

exists, where $L_{\varepsilon,\eta}$is any sub-arc about t of vanishing length. Obviously, if $\Phi(t)$ exists in the Riemann sense, it has a principal value, but the converse is not true in general.[5], p_{15}

$$\int \varphi(\tau)d\tau \qquad (20)$$

is some arbitrary constant. What is of particular interest is when z approaches a value lying on the arc L Due to the non-integrable singularity in the denominator, we must develop a means to make sense of the integral when $z = t$. This is achieved by defining the principal value of the Cauchy integral.[7], p15

The scalar problem on a closed arc L

Definition (4.3): A function $\Phi(z)$ is sectionally analytic in some set $S \subseteq C$ in which the arc L lies if it is analytic in $S \backslash L$ and continuous on L from its $\oplus$ and θ sides, taking the limiting values $\Phi^+(t)$and $\Phi^-(t)$respectively. With this definition, we proceed to state the Riemann-Hilbert problem: Given a piecewise smooth arc L, functions $G(t)$ and $g(t)$ that satisfy Holder conditions on L, with $G(t) \neq 0$ for all $t \in L$, find a sectionally analytic function $\Phi(z)$ whose limiting values from the $\oplus$ and θ sides of L satisfy

$$\Phi^+(t)= G(t)\, \Phi^-(t)+ g(t). \qquad (21)$$

The above equation is sometimes called the jump condition for $\Phi(z)$. It will be seen the stated problem always has a solution, but without any further restrictions imposed, the solution is not unique. Therefore, an extra condition that we might consider is enforcing the degree of the solution at $|z| = \infty$. This is tantamount to demanding that $\Phi(z) = z_k as\ |z| \to \infty$, for some $\kappa \in Z$. In fact, henceforth we will stipulate that $\Phi(z)$ vanishes at $|z| = \infty$, as this condition is often employed in physical problems he scalar problem on an open arc L In this section we assume L to be comprised of n non-intersecting open arcs,

$L_1, \ldots, L_n$. Every arc L will have its endpoints denoted a_j, b_j , and will be traversed from $a_j$to b_j. With these definitions made, we proceed to solve the Riemann-Hilbert problem as stated in the last. As we did previously, we begin by seeking a solution to the homogeneous

Problem $X^+(t)= G(t)\, X^-(t))$, or equivalently,

$$\log X^+(t)= \log X^+(t)= X^-(t)\log G(t). \quad (22)$$

Note that this time, however, log $G(t)$ is certainly single-valued on each of the open arcs $L_1, \ldots, L_n$., and so satisfies a Holder condition on L. We may therefore apply the Plemelj formula directly to (22), obtaining

$$\log X(z) = \frac{1}{2\pi i}\int_L^* \log\frac{G(\tau)}{z-\tau}\, d\tau,$$

$$X(z) = \exp\left[\frac{1}{2\pi i} n\sum\int_L^* \log\frac{G(\tau)}{z-\tau}\, d\tau, \qquad (23)\right.$$

But we must be mindful of the extra condition that is inherent to the problem in the case of open arcs; namely, the boundedness of solutions at the endpoints. In particular, observe that by definition $G(\tau) \neq 0$ for all $\tau \in L$. This means that $\varphi(\tau) \equiv \log G(\tau)$ is bounded everywhere on L, and in the language of $\varphi(\tau)$ belongs to the class of functions with $\gamma = 0$. Using the results of that section.[7], p22 we have, for

$$1 \le j \le n, \qquad = \frac{1}{2\pi i}\int_{L_j}^* \log G(a_j)\, d\tau \qquad \begin{Bmatrix} \frac{1}{2\pi i}\log(a_j - z), as\ z \to a_j \\ \frac{1}{2\pi i}\log(b_j - z), as\ z \to b_j \end{Bmatrix} \qquad (24)$$

and letting

$$-\frac{1}{2\pi i}\, G(a_j) = \frac{1}{2\pi}\, argG(a_j) + \log\frac{i}{2\pi}\, G(a_j) \equiv \alpha_j + iA_j \qquad (25)$$

$$\frac{1}{2\pi i}\, G(b_j) = \frac{1}{2\pi}\, argG(a_j) - \log\frac{i}{2\pi}\, G(b_j) \equiv \alpha_j + iB_j \qquad (26)$$

equation (21) for $X(z)$ implies

$$X(z)= \begin{Bmatrix} \exp\left[(\alpha_j + iA_j)\log(aj - z) + \partial_j a_j\right] = (a_j - z)^{A_j + iA_j}\, e^{\partial_j(a_j)}\ as\ z \to a_j \\ \exp\left[(\beta_j + iB_j)\log(bj + z) - \partial_j b_j\right] = (b_j - z)^{B_j + iB_j}\, e^{\partial_j b_j} as\ z \to b_j \end{Bmatrix}$$ (26).where ∂_j is the function

$$\frac{1}{2\pi i}n\ (1-\delta_{ij})\sum\int_{L}^{*} log\frac{G(\tau)}{z-\tau}d\tau \qquad (27)$$

5.The solution of scalar Riemann–Hilbert problems:
We have presented a fairly wide class of functions, the α-Hölder continuous functions, for which the limits of Cauchy integrals are well-defined and regular. We continue with the solution of the simplest RH problem on smooth, closed, and bounded curves.

5.1 Smooth, closed, and bounded curves
Problem (5.1): Find φ that solves the continuous RH problem

$$\varphi^{+}(s)\varphi^{-}(s)=f(s),\ s\in\Gamma,\ \varphi(\infty)=0\ f\in c^{0,\alpha}(\Gamma) \qquad (28)$$

where Γ is a smooth, bounded, and closed curve.
This problem is solved directly by the Cauchy integral $\varphi(z)=\vartheta_{\Gamma}f(z)$

$$\varphi^{+}(s)\varphi^{-}(s)\vartheta^{+}f(s)\vartheta^{-}f(s)=f(s),s\in\Gamma \qquad (29)$$

where Γ is a smooth, bounded, and closed curve.
To show $\varphi(\infty)=0$ we use the following lemmas, which provide more precise details.
Lemma (5.2): If $|s|^{j} f(s)\in L^{1}(\Gamma)$ *for j = 0, . . . , n, then*

$$\int_{\Gamma}^{*}\frac{f(s)}{s-z}ds=\sum_{j=1}^{n-1}c_{j}z^{-j}+\delta\,(z^{-n})\ as\ |z|\rightarrow\infty \qquad (30)$$

For $\qquad c_{j}=-\int_{\Gamma}^{*}s^{j-1}f(s)\,ds \qquad (31)$

if z, sufficiently large, satisfies inf $s\in\Gamma\,|z-s|\geq c>0.$
Lemma (5.3): For $f\in L^{1}(\Gamma)$,

$$\lim_{z\to\infty}\int_{\Gamma}^{*}\frac{f(s)}{s-z}ds=-\int_{\Gamma}^{*}f(s)ds \qquad (32)$$

where the limit is taken in a direction that is not tangential to Γ.[11], p34
We have addressed existence in a constructive way. Now we address uniqueness. Let $\psi(z)$ be another solution of Problem 5.2 The function $D(z)=\psi(z)-\varphi(z)$ satisfies $\qquad D^{+}(s)D^{-}(s)=0,\ s\in\Gamma,\ D(\infty)=0 \qquad (33)$

The trivial jump $D^{+}(s)D^{-}(s)$ is equivalent to stating that D is continuous up to Γ. It follows that D is analytic at every point on Γ. and hence D is entire. By liouville's theorem, it must be identically zero. This shows that the Cauchy integral of f is the unique solution to Problem 5.1
To show $\varphi(\infty)=0$ we use the following lemmas, which provide more precise details.

Problem (5.4): Find φ that solves the homogeneous continuous RH problem

$$\varphi^{+}(s)\varphi^{-}(s)=f(s),\ s\in\Gamma,\ \varphi(\infty)=0\ f\in c^{0,\alpha}(\Gamma) \qquad (34)$$

where Γ is a smooth, bounded, and closed curve, and $g(s)\neq 0$.
Formally, this problem can be solved via the logarithm. Consider the RH problem solved by $X(z)=\log\varphi(z)$:

$$X^{+}(s)=X^{-}(s)+G(s)\Leftrightarrow X^{+}(s)-X^{-}(s)\,G(s),\ \ G(s)=\ log\,g\,(s) \quad (35)$$

If log $g(s)$ is well-defined and Holder continuous, the solution is given by

$$\varphi(z)=exp(\vartheta_{\Gamma}G(z)). \qquad (36)$$

Furthermore, because $|\vartheta_{\Gamma}\ G(z)|<\infty$ for all $z\in C\backslash\Gamma$, and it is continuous up to Γ, we have $|\vartheta_{\Gamma}G(z)|\leq C$ for some C. This implies that $\varphi(z)$ and $1/\varphi(z)$ are both bounded, continuous functions on $C\setminus\Gamma$. To see uniqueness, let $\psi(z)$ be another solution and consider $R(z)=\psi(z)/\varphi(z)$. Then$R^{+}(s)$ =$R^{-}(s)$ on Γ, and hence $R(z)$ is entire $R(z)$ is uniformly bounded on $C\setminus\Gamma$, and thus $R(z)\equiv 1$, or $\psi(z)=\varphi(z)$.[12], p25
For a general Holder continuous function g, log g may not be well-defined. Indeed, even if one fixes the branch of the logarithm, log g generically suffers from discontinuities. To rectify this issue, we define the index of a function g with respect to traversing Γ in the positive direction to be the normalized increment of its argument:

$$ind_{\Gamma}\,g(s)\ \triangleq\frac{1}{2\pi}[arg g(s)]_{\Gamma}\ =\frac{1}{2\pi i}[log g(s)]_{\Gamma}\ =\int dlog(s) \qquad (37)$$

We defer the extension of these results to unbounded contours such as until
This case is dealt with in a more straightforward way using L p and Sobolev spaces. All solution formulae hold with slight changes in interpretation.
We move to the simplest case of a scalar RH problem with a multiplicative jump.

Problem (5.5): Find φ that solves the inhomogeneous continuous RH problem

$$\varphi^{+}(s)=\varphi^{-}(s)g(s),\ s\in\Gamma,\ \varphi(\infty)=1,\ g\in C^{0,\alpha} \qquad (38)$$

where Γ is a smooth, bounded arc extending from $z=a$ *to* $z=b$, and $g(s)$.

Problem (5.6): Find φ that solves the inhomogeneous continuous RH problem

$$\varphi^{+}(s)=\varphi^{-}(s)g(s),\ s\in\Gamma,\ \varphi(\infty)=1,\ g\in C^{0,\alpha} \quad (39)$$

where Γ is a smooth, bounded arc extending from z = a to z = b, and **g(s) ≠ 0**. Divide (3.4) by v and write

$$\frac{\varphi^{+}(s)}{V^{+}(s)}=\frac{\varphi^{-}(s)}{V^{-}(s)}+\frac{f(s)}{V^{+}(s)},\frac{\varphi(s)}{V(s)}\rho z^{n_a+n_b-1}\ as\ z\to\infty. \quad (40)$$

We assume f satisfies an α-Holder condition. Thus $f(s)/V^{+}(s)$ satisfies an (α,ζ_a)Holder condition near $z=a$ and a similar condition near $z=b$. A solution of problem, assuming possible moment conditions (13) are satisfied, is given by

$$\varphi(z)=V(z)\int_{\Gamma}^{*}\frac{f(s)}{V^{+}(s)(s-z)}ds. \quad (41)$$

we see that $\varphi(z)$ has bounded singularities at the endpoints of Γ whenever $\zeta_c\neq 0$; Otherwise, there is a logarithmic singularity present. As before,

$$\varphi(z)=V(z)\left(\int_{\Gamma}^{*}\frac{f(s)}{V^{+}(s)(s-z)}ds+p(z)\right). \quad (42)$$

is the solution, where $P(z)$ is a polynomial of degree less than (n_a+n_b) if $n_a+n_b<0$; otherwise $P=0$ and we have n_a+n_borthogonality conditions for a solution to exist with $k=n_a+n_b$Note that different fundamental solutions can be Hosen when the fundamental solution is not unique. This gives rise to solutions with different bounded ness properties.[11], p40

6.Dictation:

The fact studied will now be used solve scalar RHP. the index of function $f(x)$ with respect toγ the increment of argument in traversing:

$$Ind\,f(x)=\frac{1}{2\pi i}[\arg f(x)]_{\gamma}=\frac{1}{2\pi i}[\log f(x)]_{\gamma}=\frac{1}{2\pi i}\int_{\gamma}^{*}\log f(x) \quad (43)$$

Recall the solution of the simplest scalar RHP. let $\gamma=R$ and j_{γ}be a Holder continuous scalar function in R then the following additive scalar RHP for Y in C:

1. Y is analytic in C/R
2. $Y_{+}(t)=Y_{+}(t)+j_{Y}(x),\ x\in R$
3. $Y(z)\to 0$ as $z\to\infty$.

Has the following unique solution as Cauchy type integral

$$y(z)=C_R j_Y(z)-\frac{1}{2\pi i}\int_{R}^{*}\frac{j_Y(x)}{x-z}dx \quad (44)$$

Indeed the first condition satisfied for Cauchy type-integral definition and sokhotski-plemelj formulae for $x\in R$

$$C_R^{+}j_Y(z)=j_Y(x) \quad (45)$$

Thence $C_R j_Y(x)$ satisfied the second condition .to show the third condition we used the geometric serie in the definition of $C_R j_Y(x)$ there for

$$C_R j_Y(z)=-\frac{1}{2\pi i}\int_{R}^{*}j_Y(x)[C-\frac{1}{z})\frac{1}{1-(\frac{x}{z})}]dx \quad (46)$$

$$=\frac{1}{2\pi i}\int_{R}^{*}j_Y(x)[\sum_{n=0}^{\infty}(\frac{x}{z})^{n}]\,dx$$

$$=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}(-\frac{1}{2\pi i}\int_{R}^{*}j_Y(x)\,x^{n}dx \quad (47)$$

$$C_R j_Y(z)=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}a_n \quad (48)$$

Note that as $z\to\infty$.we have $C_R j_Y(z)\to 0$ there exists solution of $C_R j_Y(z)$,*of the* additive scalar RHP.
The scalar linear Riemann–Hilbert problem for D is stated as follows. Given
Holder continuous functions $\lambda(t)\neq f(t)$ on∂D. To find a function $\varphi(z)$ analytic in D, continuous in the closure of D with the boundary condition

$$Re\,\lambda(t)\varphi(t)=f(t),\ t\in\partial D. \quad (49)$$

This condition can be also written in the form.[8], p64

7.Results:

The Riemann-Hilbert approach has acquired wide applications in integrable systems. the scaler Riemann Hilbert problem is the function theoretical problems or finding single function.to solve we need know the first condition satisfied for Cauchy type integral definition, sokhotski-plemelj formulae for $x\in R$ and Holder Condition. The scalar linear Riemann–Hilbert problem for D is stated as follows.

8.Conclusion:

The solve of scalar Riemann–Hilbert problem for circular multiply connected domains. The method is based on the reduction of the boundary value and complex plan, hardy theorem and condition satisfied for Cauchy type integral definition and sokhotski-plemelj formulae for x. problem to a system of functional equations. In the previous works, the Riemann–Hilbert problem and its partial cases such as the Dirichlet problem were solved under geometrical restrictions to the domains. he solutions of initial value problems with both zero and nonzero initial functions are obtained and homogeneous and non-homogeneous equations are studies.

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