

INVESTIGATE A NEW TANGENT BUNDLE SPACE FROM A NEW POINT AND TWO CURVES USING MATLAB

Makarim Abdullah Mohammed Ali Mohammed¹, *Abdel Radi Abdel Rahman Abdel Gadir Abdel Rahman²

¹Lecturer , Department of Mathematics, Faculty of Education, Dongola University, Dongola, Sudan.

²Professor, Department of Mathematics, Faculty of Education, Omdurman Islamic University, Omdurman, Sudan.

***Corresponding Author:**

Abstract:

Our goal was to create a new tangent bundle space from a new point and two curves using the MATLAB program. We used the applied mathematical approach, and we arrived at an accurate model of the tangent bundle space by building an algorithm that implements the solution and displays the results and represents them graphically with ease and with high accuracy. This is what MATLAB makes a suitable working environment for dealing with tangent bundles. Given the importance of tangent bundles in modern engineering and physical sciences, this algorithm can be developed to include the most important areas of tangent bundles instead of the complex traditional method.

1. INTRODUCTION:

The tangent bundles play an important role in many fields of modern engineering, physics, technology, and astronomy. Therefore, it was necessary to highlight them and implement their solution using computing programs because they give accurate and fast results, and they are also possible to represent graphically in a multi-dimensional space instead of the traditional method. The complex was a scientific and powerful reason to study and solve it using modern mathematical methods such as MATLAB and other methods that we can also use to give accurate and fast results.

2. The Geometry of Curves

Definition (2.1): A curve in 3-space R^3 is a continuous mapping $\alpha: I \rightarrow R^3$ where I is some type of interval (e.g., $(0, 1), (a, b), [a, b], (-\infty, a], [0, I]$ etc.) on the real line R . Because the range of a is R^3 , a's output has three coordinates. We then write, for $t \in I$, a parametrization for α ,

$$\alpha(t) = (\alpha^1(t), \alpha^2(t), \alpha^3(t)) \quad (1)$$

Where the α^i are themselves functions $\alpha^i : I \rightarrow R$. A useful way to think about curves is to consider t to be time and $\alpha(t)$ to be the path of a particle in space. We say a is differentiable or smooth if each coordinate function α^i is differentiable as an ordinary real-valued function of R .

The velocity vector of α at t_0 is defined to be

$$\dot{\alpha}(t) = \left(\left. \frac{d\alpha^1}{dt} \right|_{t=t_0}, \left. \frac{d\alpha^2}{dt} \right|_{t=t_0}, \left. \frac{d\alpha^3}{dt} \right|_{t=t_0} \right) \quad (2)$$

Where $\frac{d\alpha^i}{dt}$ is the ordinary derivative and $\left. \frac{d\alpha^i}{dt} \right|_{t=t_0}$ denotes evaluation of the derivative at $t = t_0$. We shall also write $\frac{d\alpha^i}{dt}(t_0)$ for this evaluation when it is convenient. In order to interpret $\dot{\alpha}$ geometrically. [8,p(1)]

Example (2.2): The parameterized differentiable curve given by

$$\alpha(t) = (a \cos t, a \sin t, bt), t \in R,$$

Has as its trace in R^3 a helix of pitch $2\pi b$ on the cylinder $x^2 + y^2 = a^2$. The parameter t here measures the angle which the x axis makes with the line joining the origin 0 to the projection of the point $\alpha(t)$ over the xy plane (see Fig No (1))

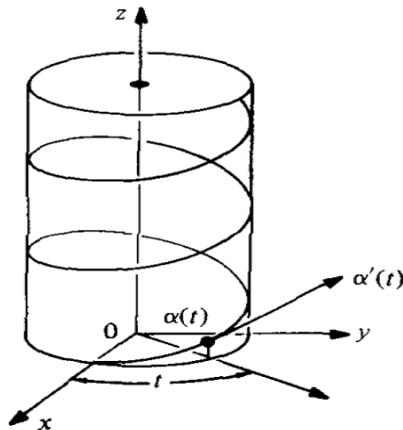


Fig No (1): The parameterized differentiable curve.

Definition (2.3): A parameterized differentiable curve $\alpha: I \rightarrow R^3$ is said to be regular if $\dot{\alpha}(t) \neq 0$ for all $t \in I$. [12,P(2)]

Definition (2.4): The velocity vector $\dot{\alpha}(t_0)$ to a at t_0 is a tangent vector to a at $\alpha(t)$. Notice that we obtain a vector with a precise length, not a line. Remember that a vector $v = (v^1, v^2, v^3) \in R^3$, has a length given by the Pythagorean theorem

$$|v| = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2} \quad (3) \quad [8,p(1)]$$

3. The Tangent Bundles

The collection of tangent spaces to a manifold has the structure of a vector bundle over the manifold, called the tangent bundle.

Definition (3.1): Let $M \subset R^k$ be a smooth m -dimensional manifold and fix a point $p \in M$. A vector $v \in R^k$ is called a tangent vector of M at p if there is a smooth curve $\gamma : R \rightarrow M$ such that $\gamma(0) = p, \gamma'(0) = v$. The set

$$T_p M := \{\gamma'(0) \mid \gamma : R \rightarrow M \text{ is smooth}, \gamma(0) = p\} \quad (4)$$

Of tangent vectors of M at p is called the tangent space of M at p (see Fig No(2)).

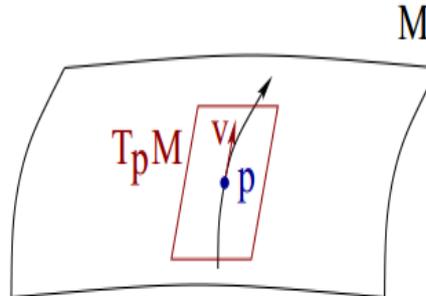


Fig No (2): The Tangent Space $T_p M$ [7, p(12)]

Definition (3.2): A differentiable curve through a point $p \in N$ is a differentiable curve $a : (-1,1) \rightarrow N$ such that $a(0) = p$. It will be denoted by $a(t)$, with $t \in (-1,1)$. When t varies in this interval, a 1-dimensional continuum of points is obtained on N . In a chart (U, ψ) around p these points will have coordinates $a^i(t) = u^i \circ \psi[a(t)]$. (5)

Definition (3.3): Let f be any differentiable real function on $U \ni p$,
 $f : U \rightarrow E^1$,

As in Figure 6.2. The vector V_p tangent to the curve $a(t)$ at point p is given by

$$V_p(f) = \frac{d}{dt} [(f \circ a)(t)]_{t=0} = \left[\frac{da^i}{dt} \right]_{t=0} = \frac{\partial f}{\partial a^i} \quad (6)$$

Notice that V_p is quite independent of f , which is arbitrary. It is, as

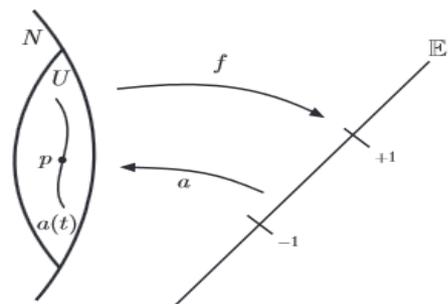


Fig No (3): A Curve Maps E^1 into N , and a Real Function Proceeds in the Converse Way. The Definition of Vector uses the Notion of Derivative on E^1 . [13, p(164)].

Definition (3.4): A tangent vector at $m \in M \subset E_r$ is a vector v in E_r of the form $v = y'(t_0)$ for some path $y = y(t)$ in M through m and $y(t_0) = m$.

Example (3.5): Let M be the surface $y_3 = y_1^2 + y_2^2$, which we parameterize by:

$$\begin{aligned} y_1 &= x^1 \\ y_2 &= x^2 \\ y_3 &= (x^1)^2 + (x^2)^2 \end{aligned}$$

This corresponds to the single chart $(U = M; x^1, x^2)$, where

$$x^1 = y_1 \text{ and } x^2 = y_2.$$

To specify a tangent vector, let us first specify a path in M , such as

$$\begin{aligned} y_1 &= \sqrt{t} \sin t \\ y_2 &= \sqrt{t} \cos t \\ y_3 &= t \end{aligned}$$

This gives the path shown in the figure.

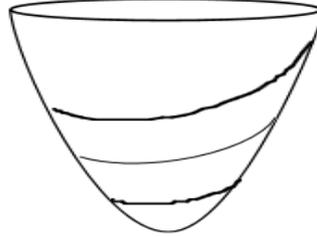


Fig No (4): A Tangent Vector Field

Now we obtain a tangent vector field along the path by taking the derivative:

$$\left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \frac{dy_3}{dt}\right) = \left(\sqrt{t} \cos t + \frac{\sin t}{2\sqrt{t}}, -\sqrt{t} \sin t + \frac{\cos t}{2\sqrt{t}}, 1\right).$$

(To get actual tangent vectors at points in M , evaluate this at a fixed point t_0 .)

Note that we can also express the coordinates x^i in terms of t :

$$x^1 = y_1 = \sqrt{t} \sin t$$

$$x^2 = y_2 = \sqrt{t} \cos t$$

Is described a path in some chart (that is, in coordinate space) rather than on the manifold itself. We can also take the derivative,

$$\left(\frac{dx^1}{dt}, \frac{dx^2}{dt}\right) = \left(\sqrt{t} \cos t + \frac{\sin t}{2\sqrt{t}}, -\sqrt{t} \sin t + \frac{\cos t}{2\sqrt{t}}\right).$$

We also think of this as the tangent vector, given in terms of the local coordinates. A lot more will be said about the relationship between the above two forms of the tangent vector below. [14, p(14)]

Definition (3.6): Let f be a differentiable real-valued function on R^3 , and let v_p be a tangent vector to R^3 . Then the number

$$v_p[f] = \frac{d}{dt}(f(p + tv))_{t=0} \quad (7)$$

Is called the derivative of f with respect to v_p .

This definition appears in elementary calculus with the additional restriction that v_p be a unit vector. Even though we do not impose this restriction, we shall nevertheless refer to $v_p[f]$ as a directional derivative.

For example, we compute $v_p[f]$ for the function $f = x^2yz$, with

$p = (1, 1, 0)$ and $v = (1, 0, -3)$. Then

$$p + tv = (1, 1, 0) + t(1, 0, -3) = (1 + t, 1, -3t)$$

Describes the line through p in the v direction. Evaluating f along this line, we get Now, hence at $t = 0$, we find $v_p[f] = -3$. Thus, in particular, the function f is initially decreasing as p moves in the v direction. The following lemma shows how to compute $v_p[f]$ in general, in terms of the partial derivatives of f at the point p . [3, p(12)]

Definition (3.7): The set of all tangent vectors at p is written as T_pM . The tangent bundle TM is the disjoint union of all the tangent spaces for all points in M .

$$TM := \bigcup_{p \in M} T_pM \quad (8) \quad [6, p(49)]$$

Definition (3.8): Let M be a C^k -manifold, let $I \subset \mathbb{R}$ be an open interval and let $\gamma: I \rightarrow M$ be a C^k -curve. for every $t \in I$, the tangent vector $\dot{\gamma}(t)$ of γ at t is an element of the tangent space $T_{\gamma(t)}M$. Hence, while t runs through I , $\gamma(t)$ runs through the tangent spaces along γ , see Fig No (3.6)

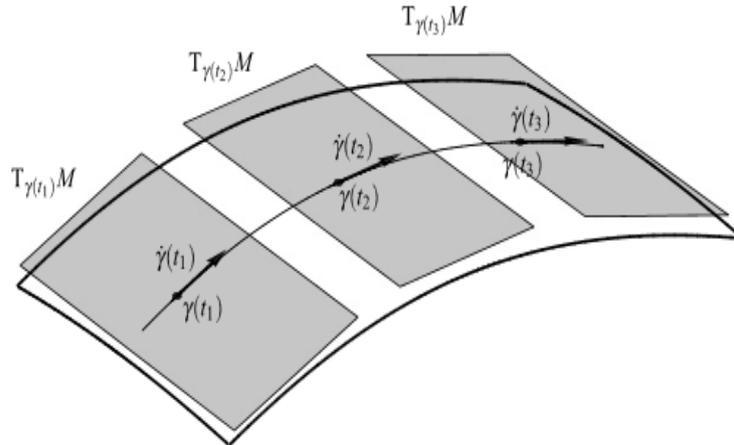


Fig No (5): Tangent Vectors Along a Curve γ in M

To follow the tangent vectors along γ it is convenient to consider the totality of all tangent spaces of M . This leads to the notion of tangent bundle of a manifold M , denoted by TM . As a set, TM is given by the disjoint union of the tangent spaces at all points of M . [5, p(53)]

Proposition (3.9): A smooth map $f: M \rightarrow N$ induces a smooth map $Df: TM \rightarrow TN$ such that:

- i. $Df(T_pM) \subset T_{f(p)}N, \forall p \in M$
- ii. The restriction to each tangent space $D_p f: T_pM \rightarrow T_{f(p)}N$ is linear: The map Df is called the differential of f , and one often uses the alternate notation $f_* = Df$. [9, p(28)]

Proposition (3.10): The tangent bundle TM of a manifold M is second countable.
Proof: Let $\{U_i\}_{i=1}^\infty$ be a countable basis for M consisting of coordinate open sets. Let ϕ_i be the coordinate map on U_i . Since TU_i is homeomorphic to the open subset $\phi_i(U_i) \times \mathbb{R}^n$ of \mathbb{R}^{2n} and any subset of a Euclidean space is second countable, TU_i is second countable. For each i , choose a countable basis $\{B_{i,j}\}_{j=1}^\infty$ for TU_i . Then $\{B_{i,j}\}_{j=1}^\infty$ is a countable basis for the tangent bundle.

Proposition (3.11): The tangent bundle TM of a manifold M is Hausdorff. [10, p(132)]

Definition (3.12): The triple (TM, M, π) is called the tangent bundle of M . TM is called the total space or the bundle manifold, M the base manifold and π the natural projection. For $m \in M, \pi^{-1}(m) \equiv T_mM$ is called the fibre over m . The vector space \mathfrak{r} is called the typical fibre and the pairs (U_α, χ_α) are called local trivializations of TM over U_α . [5, p(55)]

Example (3.13):

- i. A simple example of a nontrivial vector bundle is the infinite Möbius band M : Consider the quotient space of $\mathbb{R} \times \mathbb{R}$ by the equivalence relation $(t, s) \sim (t + 1, -s)$. The first projection gives rise to a map $p: M \rightarrow S^1$, which we claim is a nontrivial real vector bundle of rank 1 over S^1 . It is easy to see that complement of the 0-section in the total space of this bundle is connected. Therefore, the bundle cannot be the trivial bundle $S^1 \times \mathbb{R}$. Indeed, the total space of this bundle is not even homeomorphic to $S^1 \times \mathbb{R}$ but to see that needs a little bit more topological arguments.
- ii. The tangent bundle $\tau(X) = (TX, p, X)$ of any smooth submanifold $X \in \mathbb{R}^N$ is a typical example of a vector bundle of rank n , where $n = \dim X$. It satisfies the additional smoothness conditions, viz.,
 - a. Both the total space and the base space are smooth manifolds;
 - b. The projection map p is smooth and.
 - c. The homeomorphisms $\varphi: p^{-1}(U) \rightarrow U \times \mathbb{R}^n$ are actually diffeomorphisms. Over the base space B which is a smooth manifold, a vector bundle that satisfies these additional smoothness conditions will be called a smooth vector bundle. [1, p(139)]

Theorem (3.14): Let G be a Lie group. Then G is parallelizable. That is, its tangent bundle TG is trivial.

Proof:

Let $1 \in G$ denote the identity element, and T_1G the tangent space of G at 1. If G is an n -dimensional manifold, T_1G is an n -dimensional vector space. We define a bundle isomorphism of the tangent bundle TG with the trivial bundle $G \times T_1(G)$, which, on the total space level is given by a map

$$\emptyset: G \times T_1G \rightarrow TG$$

Defined as follows. Let $g \in G$. Then multiplication by g on the right is a diffeomorphism

$$\begin{aligned} \times g: G &\rightarrow G \\ x &\rightarrow xg \end{aligned}$$

Since $\times g$ is a diffeomorphism, its derivative is a linear isomorphism at every point:

$$Dg(x) : T_xG \xrightarrow{\cong} T_{xg}G.$$

We can now define

$$\emptyset: G \times T_1G \rightarrow TG$$

By

$$\emptyset(g, v) = Dg(1)(v) \in T_gG. \quad (9)$$

Clearly \emptyset is a bundle isomorphism.

Principal bundles are basically parameterized families of topological groups, and often Lie groups. In order to define the notion carefully we first review some basic properties of group actions. Recall that a right action of topological group G on a space X is a map

$$\begin{aligned} \mu : X \times G &\rightarrow X \\ (x, g) &\rightarrow xg \end{aligned}$$

Satisfying the basic properties

- i. $x \cdot 1 = x$ For all $x \in X$
- ii. $(x \cdot g_1)g_2 = (xg_1)g_2$ For all $x \in X$ and $g_1, g_2 \in G$. [13, p(8)]

Definition (3.15): A vector field X on a manifold M is a function that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi: TM \rightarrow M$ and the vector field is smooth if it is smooth as a map from M to TM . [11, p(136)]

Definition (3.16): A map $\xi: M \rightarrow TM$ is called a vector field on M , if for all $p \in M$ we have $\pi(\xi(p)) = p$. (10) [4, p(33)]

4. Method and discussion:

Example (4.1): Let $S = \emptyset(U)$, where $U = \{(r, \theta) \mid 0 < r < 2, -\pi < \theta < \pi\}$ and $\emptyset: U \rightarrow R^3$ is given by $\emptyset(r, \theta) = (r \cos \theta, r \sin \theta, r^2)$.

We investigate the tangent space to S at the point $p_1 = \emptyset_1(\sqrt{2}, \frac{\pi}{4}) = (1, 1, 2)$.

We first consider two curves $c_i: I_i \rightarrow U, i = 1, 2$, where $I_1 = (-\frac{1}{2}, \frac{1}{2})$,

$I_2 = (-\frac{\pi}{2}, \frac{\pi}{2})$, Given by

$$c_1(t) = (t + \sqrt{2}, \frac{\pi}{4}),$$

And

$$c_2(t) = (\sqrt{2}, t + \frac{\pi}{4}).$$

Geometrically, the image curves $c_1(I_1)$ and $c_2(I_2)$ are parallel to the coordinate axes in U and intersect at $(\sqrt{2}, \frac{\pi}{4}) = c_1(0) = c_2(0)$.

Now define $c_1: I_1 \rightarrow S$ and $c_2: I_2 \rightarrow S$ by $c_1 = \emptyset \circ c_1$ and $c_2 = \emptyset \circ c_2$.

Explicitly,

$$c_1(t) = \left((t + \sqrt{2})\cos\left(\frac{\pi}{4}\right), (t + \sqrt{2})\sin\left(\frac{\pi}{4}\right), ((t + \sqrt{2}))^2 \right)$$

$$= \left(\frac{t}{\sqrt{2}} + 1, \frac{t}{\sqrt{2}} + 1, (t + \sqrt{2})^2 \right),$$

$$c_2(t) = \left(\sqrt{2}\cos\left(t + \frac{\pi}{4}\right), \sqrt{2}\sin\left(t + \frac{\pi}{4}\right), 2 \right)$$

$$= (\cos t - \sin t, \cos t + \sin t, 2)$$

The image curves $c_1(I_1)$ and $c_2(I_2)$ are, respectively, the curve of intersection of the plane $x = y$ with the paraboloid $z = x^2 + y^2$ and the curve of intersection of the plane $z = 2$ with the paraboloid $z = x^2 + y^2$.

Note that both $c_1(0) = p$ and $c_2(0) = p$, so that the tangent vectors $v_{p_1} = \dot{c}_1(0)$ and $w_{p_1} = \dot{c}_2(0)$ are by definition elements of $T_p(S)$. Explicitly,

$$v_{p_1} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2\sqrt{2} \right\rangle_p$$

$$= \frac{1}{\sqrt{2}} \langle 1, 1, 4 \rangle_p$$

And

$$w_{p_1} = \langle -1, 1, 0 \rangle_p$$

In fact, we show that every linear combination of v_{p_1} and w_{p_1} is a tangent vector to S at p_1 .

We note that the vectors $v_{p_1} = \frac{1}{\sqrt{2}} \langle 1, 1, 4 \rangle_p$ and $w_{p_1} = \langle -1, 1, 0 \rangle_p$ studied above are two linearly independent vectors in the 2dimensional subspace W_1 , and so

$$T_{p_1}(S) = W_1 = \text{Span} \{v_{p_1}, w_{p_1}\} \quad *$$

Second, we also check the tangent space to S at the point $p_2 = \phi_2(\sqrt{3}, \frac{\pi}{3}) = (\frac{\sqrt{3}}{2}, \frac{3}{2}, 3)$.

We second consider two curves $\tilde{c}_i : I_i \rightarrow U, i = 3, 4$, where $I_3 = (\frac{-1}{2}, \frac{1}{2})$, $I_4 = (\frac{-\pi}{2}, \frac{\pi}{2})$, Given by

$$\tilde{c}_3(t) = (t + \sqrt{3}, \frac{\pi}{3}),$$

And

$$\tilde{c}_4(t) = (\sqrt{3}, t + \frac{\pi}{3}).$$

The image curves $\tilde{c}_3(I_3)$ and $\tilde{c}_4(I_4)$ are parallel to the coordinate axes in U and intersect at $(\sqrt{3}, \frac{\pi}{3}) = \tilde{c}_3(0) = \tilde{c}_4(0)$.

Now define $c_3 : I_3 \rightarrow S$ and $c_4 : I_4 \rightarrow S$ by $c_3 = \phi \circ \tilde{c}_3$ and $c_4 = \phi \circ \tilde{c}_4$.

$$c_3(t) = \left((t + \sqrt{3})\cos\left(\frac{\pi}{3}\right), (t + \sqrt{3})\sin\left(\frac{\pi}{3}\right), ((t + \sqrt{3}))^2 \right)$$

$$= \left(\frac{t + \sqrt{3}}{2}, \frac{t + \sqrt{3}}{2}\sqrt{3}, (t + \sqrt{3})^2 \right)$$

$$c_4(t) = \left(\sqrt{3}\cos\left(t + \frac{\pi}{3}\right), \sqrt{3}\sin\left(t + \frac{\pi}{3}\right), (\sqrt{3})^2 \right)$$

$$c_4(t) = \left(\sqrt{3}(\cos(t)\cos\left(\frac{\pi}{3}\right) - \sin(t)\sin\left(\frac{\pi}{3}\right)), \sqrt{3}(\sin(t)\cos\left(\frac{\pi}{3}\right) + \cos(t)\sin\left(\frac{\pi}{3}\right)), 3 \right)$$

$$= \left(\frac{\sqrt{3}}{2}(\cos(t) - \sin(t)), \frac{\sqrt{3}}{2}(\sin(t) + \cos(t)), 3 \right)$$

Note that both $c_3(0) = p_2$ and $c_4(0) = p_2$, so that the tangent vectors $H_{p_2} = \dot{c}_3(0)$ and $Q_{p_2} = \dot{c}_4(0)$ are by definition elements of $T_p(S)$. Then

$$H_{p_2} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 2\sqrt{3} \right\rangle_p$$

$$Q_{p_2} = \left\langle \frac{-\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle_p$$

Studied above are two linearly independent vectors in the 2dimensional subspace W_2 , and so

$$T_{p_2}(S) = W_2 = \text{Span} \{H_{p_2}, Q_{p_2}\} \quad **$$

Investigate the tangent bundle space on the tangent spaces $T_p(S)$ for all point in S , where the tangent bundle TS is the disjoint union of all the tangent spaces for all points in S .

$$TS = \bigcup T_p S$$

From * and ** then

$$TS = \{T_{p_1}(S), T_{p_2}(S)\}$$

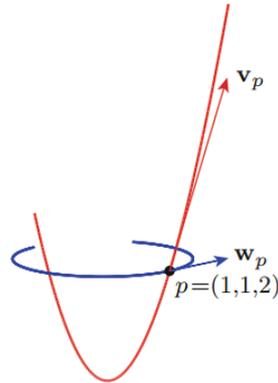


Fig No (6): The Tangent Space $T_{p_1}(S) = W = \text{Span}\{v_{p_1}, w_{p_1}\}$ [2, p(84)]

(4.2)Solution by using Matlab:

```
clear all
clc
syms S pi U r th R I1 I2 i p C1 C2 C3 C4 t Vp1 Wp1 Hp2 Qp2 x1 y1 z1 x2 y2 z2
pi=180;
r =linspace(5,20,10);
th =linspace(-pi,pi,10);
t=0;
% (t,?) = (????,???? ?,?)
I1=linspace(-0.5,0.5,10);
I2=linspace(-pi/2,pi/2,10);
C1= [t+sqrt(2) pi/4];
C2= [sqrt(2) t+(pi/4)];
C1= [(t/sqrt(2))+1 (t/sqrt(2))+1 (t+sqrt(2))^2]
C2= [(cos(t)-sin(t)) (cos(t)+sin(t)) 2]
t=0;
x1=((1/sqrt(2))*t)+1;
x1=diff(x1);
y1=((1/sqrt(2))*t)+1;
y1=diff(y1);
z1=(t+sqrt(2))^2;
z1=diff(z1);
x2= cos(t)-sin(t);
x2=diff(x2);
y2= cos(t)+sin(t);
y2=diff(y2);
z2=2;
z2=diff(z2);
% x1(t)=x1
% y1(t)=y1
% z1(t)=z1
% x2(t)=x2
% y2(t)=y2
% z2(t)=0
% Vp1=[x1 y1 z1]
% Wp1=[x2 y2 z2]
Vp1=[1/sqrt(2) 1/sqrt(2) 2*sqrt(2)]
Wp1=[-1 1 0]
subplot(2,1,1)
plot(Vp1,Wp1)
xlabel('Vp1');
ylabel('Wp1');
title('Combination of Vp1 and Wp1 is a Tangent Vector to S at p1')
```

```

% We also check the tangent space to ? at the point ?2
I1=linspace(-0.5,0.5,10);
I2=linspace(-pi/2,pi/2,10);
C3= [t+sqrt(3) pi/3];
C4= [sqrt(3) t+(pi/3)];
C3= [(t+sqrt(3))*cos(pi/3) (t+sqrt(3))*(sin(pi/3)) (t+sqrt(3))^2]
C4= [(sqrt(3))*cos((t+(pi/3))) (sqrt(3))*sin((t+(pi/3))) sqrt(3)^2]
x1=(t+sqrt(3))*cos(pi/3);
x1=diff(x1);
y1=(t+sqrt(3))*(sin(pi/3));
y1=diff(y1);
z1=(t+sqrt(3))^2;
z1=diff(z1);
x2= (sqrt(3))*cos((t+(pi/3)));
x2=diff(x2);
y2= (sqrt(3))*sin((t+(pi/3)));
y2=diff(y2);
z2=sqrt(3)^2;
z2=diff(z2);
% Hp2=[x1 y1 z1]
% Qp2=[x2 y2 z2]
Hp2=[0.5 sqrt(3)/2 2*sqrt(3)]
Qp2=[-sqrt(3)/2 sqrt(3)/2 0]
subplot(2,1,2)
plot(Hp2,Qp2,'r')
xlabel('Hp2');
ylabel('Qp2');
figure
hold on
plot(Vp1,Wp1);
plot(Hp2,Qp2,'r');
xlabel('Vp1 & Hp2');
ylabel('Wp1 & Qp2');
title('Disjoint Union of all the Tangent Spaces for all Points in S')
hold off

```

Result:

```

C1 = 1.0000  1.0000  2.0000
C2 = 1  1  2
Vp1 = 0.7071  0.7071  2.8284
Wp1 = -1  1  0
C3 = -1.6496 -0.5279  3.0000
C4 = -1.6496 -0.5279  3.0000
Hp2 = 0.5000  0.8660  3.4641
Qp2 = -0.8660  0.8660  0

```

Represent the Solution Graphically

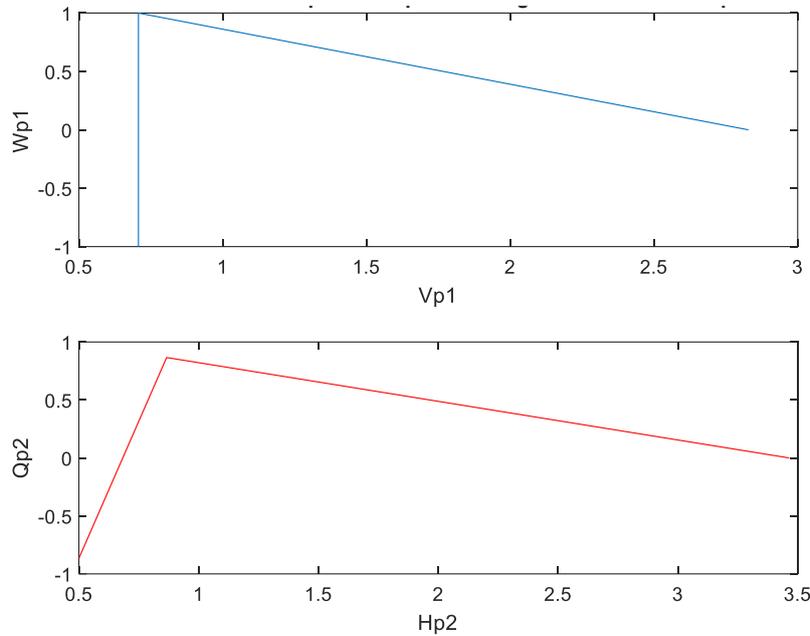


Fig No (7): Combination of v_{p_1} and w_{p_1} is a Tangent Vector to S at p_1 , H_{p_2} and Q_{p_2} is a Tangent Vector to S at p_2

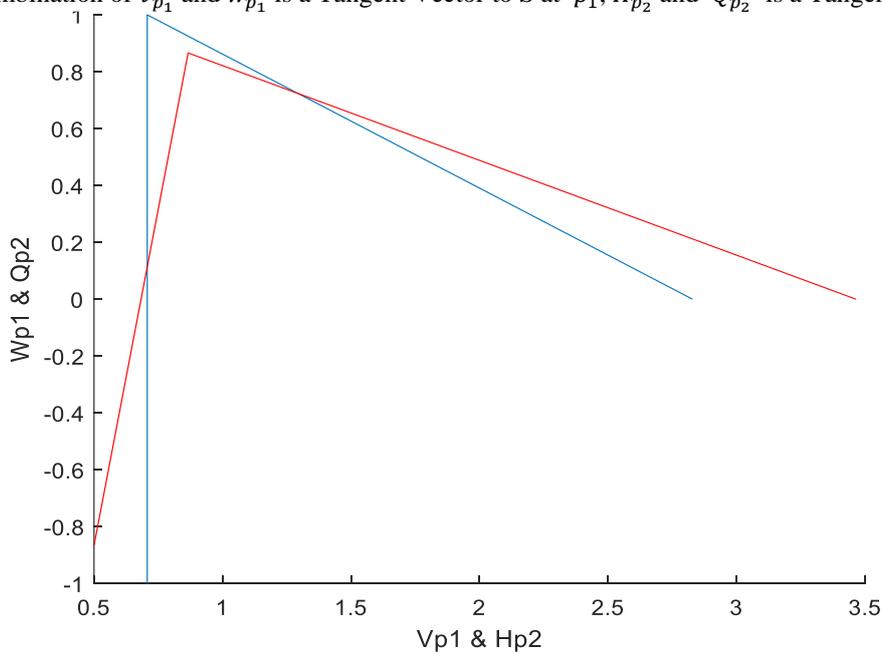


Fig No (8): Disjoint Union of all the Tangent Spaces for all Points in S

(4.3)Discuss the Results:

We discussed evaluating the performance of the computerized solution for tangent bundles, where we used a modern mathematical method, which consists of building a powerful algorithm that implements the solution, extracts the results, and represents them graphically, and it gave excellent results. In this study, we focused on the computerized solution to obtain accurate results, while other studies focused on the traditional method of dealing with tangent bundles. Through this study, we achieved great success in creating a tangent bundles from two spaces based on 4 curves (C_1, C_2, C_3, C_4), On the two points (p_1, p_2) in a two-dimensional spaces, while we were not able to apply it in a three-dimensional space, as we were satisfied with only the graphical representation of the tangent bundles , and the model achieved an accuracy of no less than 95% in solving the tangent bundles on the MATLAB program, which makes the MATLAB program It is suitable to be a suitable working environment in dealing with tangent bundles due to their importance in modern engineering sciences, artificial intelligence, and simulation programs.

Results:

After implementing all the steps in the previous example (4.1) on the MATLAB program by building a smooth algorithm that allows processing the largest number of points on space, a model was obtained automatically during the implementation and testing process that gave accurate and fast results. This model shows the graphical representation of the tangent bundle formed over space after performing the differentiation process on the curves resulting from

substituting point's p_1 and p_2 . The optimal solution is the tangent bundle, and many points can be added to this algorithm to form an expanded bundle over space.

Conclusion

We presented an accurate model of tangent bundles using the MATLAB program, by applying all the steps mentioned in Example (4.1) by building an algorithm that allows the solution to be implemented and the results presented and represented graphically. This makes the MATLAB program a suitable working environment for dealing with tangent bundles in Different fields.

References:

- [1]. Anant R.Shastri, Element Differential Topology, United States of America, CRC Press, 2011, p(121-141)
- [2]. Andrew McInerney, First Steps in Differential Geometry Riemannian, Contact, Symplectic, New York, Springer, 2013, p(39-111).
- [3]. Barrett O'Neill, Elementary Differential Geometry, Second Edition, the United States of America, Elsevier Inc, 2006, p (12-195) .
- [4]. Christian Bär, Differential Geometry, Potsdam, 3D-XplorMath and gimp, 2013, p(33).
- [5]. Gerd Rudolph, Matthias Schmidt, Differential Geometry Mathematical Physics, New York, Springer, 2013, p (53-93).
- [6]. Jeffrey Marc Lee, Differential Geometry Analysis and Physics, 2000, p(15-57).
- [7]. Joel W. Robbin , Dietmar A. Salamon, Introduction to Differential Geometry, 2011,p(12).
- [8]. John Oprea , Differential Geometry and its Applications, Second Edition , Washington, The Mathematical Association of America, 2007, p(400-401).
- [9]. Liviu I Nicolaescu, Lectures on the Geometry of Manifolds, Third Edition, Singapore, World Scientific, 2021, p(28)
- [10]. Loring W. Tu, An Introduction to Manifolds, Second edition, New York, Springer, 2011, p(132).
- [11]. Loring W. Tu, Differential Geometry: Connections, Curvature, and Characteristic Classes, Switzerland, Springer, 2017,p(129-254).
- [12]. Mantredo P. do carmo, Differential geometry of curves and surfaces, New Jersey, Prentice-Hall, Inc., 1976, p(180)
- [13]. Relph L.cohen, The Topology of Fiber Bundles Lecture Notes, California, Stanford- university, 1998, p(8,9).
- [14]. Stefan Waner, Introduction to Differential Geometry and General Relativity, third printing, New York, Departments of Mathematics and Physics, Hofstra University ,2002,p(14-20).