ON SOME MANIFOLDS OF CONSTANT NEGATIVE CURVATURE

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Abstract
Manifold of constant negative curvature a great role in the field of physics, mathematics and engineering because it paves to the knowledge Gaussian curvature, n-sphere(s) is a topological n-manifold and objects of constant negative curvature are less familiar but they do appear in nature in the shape of corals and leave. No surprisingly, its plays an important role in geometric topology. The study aims to explain a generalization manifold of constant negative curvature. WE followed the analytical induction mathematical method. We found the following some results. Manifold of constant negative curvature indicates to know the behavior of some of the functions and also it reveals the Cartan-Hadamard theorem which is considered one of the importance aims of simply-connected manifold of nonpositive sectional curvature.

Keyword: *Surfaces, Manifolds, Topological Manifolds, Constant Negative Curvature.*

1. INTRODUCTION:

We know that any two point in a connected, simply connected complete, manifold M of constant negative curvature can be connected by a unique geodesic. Thus, the entire manifold M is geodetically convex and its infectivity radius is infinity. This continues to hold in much greater generality for manifolds with negative curvature, [1], pp1.

We studied the geometric notion of a differential system describing manifold of a constant negative curvature. Constant curvature metrics surround us; we live in Euclidean space of zero curvature little soap bubbles have positive constant curvature. Objects of constant negative curvature are familiar, but they do appear in Nature in the shape of corals and leaves. No surprisingly, constant curvature play an important role in geometric topology, which studies manifolds. For example, of a constant negative curvature it can be show that a surface of revolation of constant Gaussian curvature. Then we proved that the most important local-global theorem of manifold constant negative curvature is the Cartan-Hadamard theorem ,topologically characterizes complete, simply-connected manifold with negative curvature : they are diffeomorphic to $\mathbb{R}$.

2. Manifolds:

The core idea of both differential geometry and modern geometrical dynamics lies under the concept of manifold. Manifold is an abstract mathematical space, which locally resembles the space described by Euclidean geometry, but which globally may have a more complicated structure. A manifold can be constructed by 'gluing' separate Euclidean space together, for example, a world map can be made by gluing many maps of local regions together and accounting for the resulting distortion. Therefore, the surface of earth is manifold, locally it seems to be flat, but viewed as a whole from the outer space (globally) it is actually round. Another example of a manifold of a constant negative curvature is a circle, small piece of a circle appears to be like a slightly bent part of a straight line segment but overall the circle and the segment are different one-dimensional manifold, [6], pp2.

A manifold is an abstract mathematical space in which every point has a neighbor- hood which resembles Euclidean space, but in which the global structure may be more complicated. In discussing manifold, the idea of dimension is important. For example, lines are one-dimensional and planes tow-dimensional. In one-dimensional manifold (or one-manifold), every point has a neighborhood that looks like a segment of a line. Examples of one-manifolds include a line, a circle, and two separate circles. in a tow-manifold, every point has a neighborhood that look like a disk. Examples include a plane, the surface of a sphere and the surface of a torus. Manifold are important objects in mathematics and physics because they allow more complicated structure s to be expressed and understood in terms of the relatively well-understood properties of simpler spaces. [6], pp2

Definition (2.1): [6], pp2.

A topological space X is said to be Hausdorff if for any two distinct points $xy \in X$ there exist disjoint open sets U and V with $x \in U$ and $y \in V$.

The study of manifolds combines many important areas of mathematics: it generalizes concepts such as curves with the ideas from linear algebra and topology. Certain special classes of manifolds also have additional algebraic structure; they may behave like groups, for instance. An atlas describes how a manifold is glued together from simpler pieces where each piece is given by a chart (coordinate chart or local coordinate system). The description of most manifold requires more than one chart. An atlas is not unique as all manifolds can be covered multiple ways using different combinations of charts. In a one-dimensional manifold (or one-manifold), every point has a neighborhood that looks like a segment of line. In a two-dimensional, every point has a neighborhood that looks like a disk. Examples include a plane, the surface of a sphere and the surface of a torus, [10], pp1.

Definition (2.2):[6],pp2.

An atlas A on a manifold M is said to be maximal if for any compatible atlas A' on M any coordinate chart $(x, U) \in A'$ is also a member of A.

This definition of atlas is exactly analogous to the non– mathematical meaning of atlas. Each individual map in an atlas of the world gives a neighborhood of each point on the globe that is homeomorphic to the plane. While each individual map does not exactly line up with other maps that it overlaps with, the overlap of two maps can still
be compared. Manifolds are Euclidean Spaces. Different choices for simple spaces and compatibility conditions give different objects. The dimension of the manifold at a certain point is the dimension of the Euclidean space charts at that point map to (number in the definition), . All points in a connected manifold have the same dimension. In topology and related branches of mathematics, a connected space is a topological space which cannot be written as disjoint union of two or more nonempty spaces. connectedness is one of the principal topological properties that is used to distinguish topological spaces. A manifold with empty boundary is said to be closed manifold if it is compact, and open manifold if it is not compact. All one-dimensional manifolds are curves and all two-dimensional manifolds are surfaces, [6], pp2.

Definition (2.3): [6] , pp 2.

A manifold is a Hausdorff space M with a countable basis such that for each point $p \in M$ there is a neighborhood U of p that is homeomorphic to $\mathbb{R}^n$ for some integer n. If the integer n is the same for every point in M, then M is called a n-dimensional manifold.

We are almost ready to give the official definition of manifolds. We need just one more preliminary definition, which captures in a precise way the intuitive idea that a manifold should look “locally” like Euclidean space. A topological space M is said to be locally Euclidean of dimension$n$ if every point of$M$ has a neighborhood in Mthat is homeomorphic to an open subset of $\mathbb{R}^n$. For some purposes, it is useful to be more specific about the kind of open subset we use to characterize locally Euclidean spaces. The next lemma shows that we could have replaced "open subset" by open ball or by $\mathbb{R}^n$ itself.

Topological Manifolds:

Lemma (2.4): [3], pp38.

A topological space M is locally Euclidean of dimension n if and only if either of the following properties holds:
(a) Every point of$M$has a neighborhood homeomorphic to an open ball in R^n.
(b) Every point of$M$has a neighborhood homeomorphic to $\mathbb{R}^n$.

Definition (2.5): [4], pp (2-3). Suppose M is a topological space. We say that M is a topological manifold of dimension n or a topological n-manifold if it has following

i. M is a Haausdorff space :for every pair of distinct points $p, q \in M$, there are disjoint open subset $U, V \subset M$,such that $p \in U$and $q \in V$.

ii. M is second-countable: there exists a countable basis for the topology of M.

iii. M is local Euclidean of dimension n each point of M has neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$ .The third properties means more specifically , that for each $\in M$ can find

a. open subset $U \subset M$containing p,
b. an open subset$\widehat{U} \subset R^n$ and
c. a homeomorphism $\varphi: U \rightarrow \widehat{U}$.

Definition (2.6): [11], pp8. The simplest kind of manifold to define is the topology manifold , which looks locally like some "ordinary " Euclidean space $\mathbb{R}^n$. Formally, a topological manifold is a topological space locally homeomorphic to a Euclidean space. This means that every point has a neighborhood for which there exists a home- omorphism(a bijective continuous function whose inverse is also continuous) mapping that neighborhood to $\mathbb{R}^n$. These homeomorphisms are the chart of the manifold. Usually additional technical assumptions on the topological space are made to exclude pathological cases. It is customary to require that the space be Hausdorff and second countable. The dimension of the manifold at a certain point is the dimension of the Euclidean space charts that point map to (number *n* in the definition). All points in a connected manifold have the same dimension. Some authors require that all charts of a topological manifold map to the same Euclidean space. In That case every topological manifold has a topological invariant, its dimension. Other authors allow disjoint unions of topological manifolds with differing dimensions to be called manifolds.

Example (2.7): [4], pp (5-6). For each integer $n \leq 0$,the unit *n-sphere* $S^n$is Hausdorff and second- countable because it is a topological subset of R^{n+1}. To show that it is locally Euclidean, for each index $i = 1, \ldots \ldots, n+1$ let U_i^+denote the subset of $R^{b=n+1}$ where the i the coordinate is positive:

$$U_i^+ = \{(x^1, \ldots.., x^n) \in R^{n+1}: x^i > 0\}. \quad (1)$$

(See Figure No.1) similarly, U_i^- is the set where $x^i < 0$. Let $f: B^n \rightarrow R$ be the conti- nuous function

$$f(u) = \sqrt{1 - |u|^2}.$$

Then for each $i = 1, \ldots \ldots, n+1$, it is easy to check $U_i^+ \cap S^n$ is the graph of the function

$$x^i = f(x^i, \ldots.., \widehat{x^i}, \ldots.., x^{n+1}),$$

Where the hat indicates that $x^i$is omitted. Similarly $U_i^- \cap S^n$is the graph of the

$$x^i = -f(x^i, \ldots.., \widehat{x^i}, \ldots.., x^{n+1}).$$

Thu, each subset $U_i^{\pm} \cap S^n$ is locally Euclidean of dimension n, and the maps $\varphi_i^{\pm}: U_i^+ \cap S^n \rightarrow B^n$given by

$$\varphi_i^{\pm}(x^i, \ldots.., x^{n+1}) = x^i(x^i, \ldots.., \widehat{x^i}, \ldots.., x^{n+1}),$$

are graph coordinates for S^n. Since each point of $S^n$is in the domain of at least one of these $2n+2$ chart, S^n is a topological n −manifold.

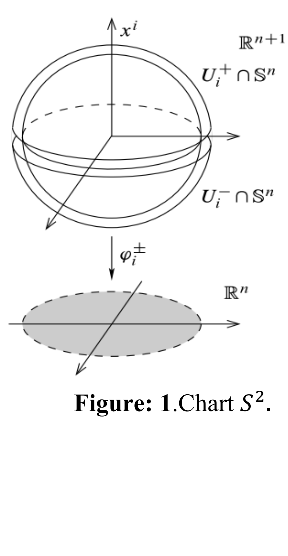

Figure: 1.Chart S^2.

Example (2.8): [4], pp7. Suppose $M_1, \ldots, M_k$are topological manifolds of dimensional $n_1, \ldots, n_k$, respectively. The product space $M_1 \times \ldots \times M_k$is shown to be a topological manifold of dimensional $n_1 + \cdots + n_k$ as follows. It is Hausdorff and second –countable, so only the locally Euclidean property needs to be ckecked. Given any point $(p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k$,we can chose a coordinate chart $(U_i, \varphi_i)$for each $M_i$with $p_i \in U_i$. The product map

$$\varphi_1 \times \ldots \times \varphi_k : U_1 \times \ldots \times U_k \to \mathbb{R}^{n_1 + \cdots + n_k}$$

is a homeomorphism onto its image, which is a product open subset of $\mathbb{R}^{n_1+\cdots+n_k}$. Thus, $M_1 \times \ldots \times M_k$is a topological manifold of dimensional $n_1 + \cdots + n_k$, with chart of form $(U_1 \times \ldots \times U_k, \varphi_1 \times \ldots \times \varphi_k)$.

i. Riemannian Manifolds:

Definition (2.9): [11], pp8. To measure distances and angles on manifolds, manifold must be Riemannian. A Riemannian manifold is an analytic manifold in which each tangent space is equipped with an inner product $\langle -,- \rangle$ in a manner which varies smoothly from point to point . Given two tangent vectors u and v the inner product $\langle u, v \rangle$ gives a real number .The dot (or scalar) products a typical example of an inner product . This allows one to define various notion such as length, angles, areas (or volumes), curvature, gradients of functions and divergence of vector fields. Most familiar curves and surfaces, including n-sphere and Euclidean space can be given the structure of a Riemannian manifold.

ii. Smooth Manifold:

Definition (2.10): [4], pp13. A smooth manifold is a pair $(M, \mathcal{A})$,where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Example (2.11): [3], pp17. A topological manifold M of dimension 0 is just a countable discrete space. For each point $p \in M$; the only neighborhood of p that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$itself, and there is exactly one coordinate map $\varphi: \{p\} \to \mathbb{R}^0$. Thus, the set of all charts on M trivially satisfies the smooth compatibility condition, and each 0-dimensional manifold has a unique smooth structure.

Example (2.12): [4], pp22. Let V an n-dimensional real vector space .For any integer $0 \le k \le n$, we let$G_k(V)$. denote the set of all k-dimensional linear subspace of V. We will show that $G_k(V)$ can be naturally given the structure of a smooth manifold of dimension $k(n-k)$. With this structure, it is called Grassmann manifold or simply a Grassmannian. In the special case $V= R^n$, the Grassmannian $G_k(R^n)$is often denoted by some simpler notation such as $G_{k,n}$, or $=G(k,n)$. Note that $G_1(R^{n+1})$ is exactly the n-dimensional projective space R^n.

Example (2.13): [4], pp17. For each nonnegative integer n, the Euclidean space $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlasconsiting of the single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$. We call this the standard smooth structure on$\mathbb{R}^n$ and the resulting coordinate map standard coordinates. Unless we explicitly specify otherwise, we always use this smooth structure on $\mathbb{R}^n$. With respect to this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U, φ)such that $\varphi$is a diffeomorphism (in the sense of ordinary calculus) from U to another open subset $\widehat{U} \subseteq \mathbb{R}^n$.

v. Differentiable Manifolds: [11]

For most applications a special kind of topological manifold, a differentiable manifold. If the local chart on a manifold are compatible in a certain sense, one can define directions, tangent spaces and differentiable functions on that manifold. In particular, it is possible to use calculus on a differentiable manifold. Each point of an n-dimensional differentiable manifold has a tangent space through the point.

Two important classes of differentiable manifolds are smooth and analytic manifolds. For smooth manifolds the transition maps are smooth, that is infinitely differentiable. Analytic manifolds are smooth manifolds with the additional condition that the transition maps are analytic (a technical definition which loosely means that Taylor's theorem holds). The sphere can be given analytic structure, as can most familiar curves and surfaces.

3. Constant Negative Curvature:

The definition of constant negative curvature it can be shown that a surface of revolution of constant Gaussian curvature, the standard definition can easily be discussed by using the following example

Example (3.1): [2], pp (196-198). A surface with constant negative Gaussian curvature, however, we have to construct a new surface. To this end, we examine again the surface of revolution

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)) \qquad (2)$$

Obtained by rotating the unit- speed curve$u \to (f(u), 0, g(u))$ in the $xz - plane$ around the $z - axis$. We have the Gaussian curvature is

$$K = \frac{LN - M^2}{EG - F^2} \qquad (3)$$

in Equation(2)where ,we can assume that $f > 0$ and $f^2 + g^2 = 1$, everywhere (a dot denoting d/du). We found that

$$E = 1, F = 0, G = f^2, L = f\dot{g}, \quad M = 0, N = f\dot{g}.$$

So, the Equation (3) becomes

$$K = \frac{(\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{f}\dot{g}}{f^2}. \quad (4)$$

We can simplify this formula by noting that $\dot{f}^2 + \dot{g}^2 = 1$ implies(by differentiating with respect to u) that $\dot{f}\ddot{f} + \dot{g}\ddot{g} = 0$,

$$\therefore (\dot{f}\ddot{g} - \ddot{f}\dot{g})\dot{g} = -\dot{f}^2\ddot{f} - \ddot{f}\dot{g}^2 = -\ddot{f}(\dot{f}^2 + \dot{g}^2) = -\ddot{f},$$

$$\therefore K = -\frac{\ddot{f}f}{f^2} = -\frac{\ddot{f}}{f}. \quad (5)$$

Suppose first that $k = 0$, everywhere.Then the Equation(5) gives $\ddot{f} = 0,\ so$ $f(u) = au + b$for some constant $a\ and\ b$.Since $\dot{f}^2 + \dot{g}^2 = 1$, we get $\dot{g} = \pm\sqrt{1 - a^2}$(so we most have $|a| \le 1$)and hence $g(u) = \pm\sqrt{1 - a^{2u}} + c$, where c is another constant. By applying a translation along the $z - axis$ we can assume that $c = 0$, and by applying a rotation by π about the $x - axis$, if necessary , we can assume that the sign is +. This gives the ruled surface

$$\sigma(u, v) = (b\cos v, b\sin v, 0) + u\left(a\cos v, \quad a\sin v, \sqrt{1 - a^2}\right). \quad (6)$$

If $a = 0$ that is a circular cylinder; if $|a|$=1it is the $xy - plane$; and if 0< $|a| < 1$ it is a circular cone (to see this put $\tilde{u} = au + b$).

Now suppose that $K > 0$, say $K = 1/R^2$, where $R > 0$ is a constant. Then, Equation (5) becomes

$$\ddot{f} + \frac{f}{R^2} = 0; \quad (7)$$

which has the general solutions

$f(u) = a\cos\left(\frac{u}{R} + b\right)$,where $a\ and\ b$ are constant. We can assume that $b = 0$ by performing a reparmetrization $\breve{u} = u + Rb, \breve{v} = v. Then,$ up to a change of sign and adding a constant,

$$g(u) = \int \sqrt{1 - \frac{a^2}{R^2}\sin\frac{u}{R}}\, du. \quad (8)$$

The integral in the formula for $g(u)$ can be evaluated in terms of 'elementary' functions only when $a = 0$ or $\pm R$. The case $a = 0$ does not give a surface , if $a = R$, then $f(u) = R\cos\frac{u}{R}$, $g(u) = R\sin\frac{u}{R}\ and$ we have a sphere of radius $R(the\ case\ a = -R$ can be reduced to this by rotating the surface by πaround the $z - axis$). Suppose finally that $K < 0$. We can restrict ourselves to the case $K = -1$, as the general case can be obtained from this by applying a dilation of R^3.In view preceding case , we think of a surface with $K = -1$ as a 'sphere of imaginary radius ' $\sqrt{-1}$, or 'pseudosphere'.

When $K = -1$ the general solution of Equation (5) is

$$f(u) = ae^u + be^{-u}, \quad (9)$$

where a and b are arbitrary constant. The function $g(u)$ can be expressed in terms of elementary function only if one a or b is zero. If $b = 0$ we can assume that $a = 1$ by a reparametrization $u \to u +$ constant and the case in which $a = 0$ can be reluced to case $b = o, u \to u -$. Suppose then that $a = 1\ and\ b = 0$, then ,$f(u) = e^u$ and we can take

$$g(u) = \sqrt{1 - e^{2u}}du \quad (10)$$

Note that we must have $u \le 0$ for the integral in Equation (10) to make sense, since otherwise $1 - e^{2u}$would be negative. The integral can be evaluated by putting $\cos\theta = e^u$.Then

$$\int \sqrt{1 - e^{2u}}du = -\int \frac{\sin^2\theta}{\cos\theta}d\theta = \sin\theta - \ln(\sec\theta + \tan\theta)$$
$$= \sqrt{1 - e^{2u}} - \ln(e^{-u} + \sqrt{e^{-2u} - 1}). \quad (11)$$

We have omitted the arbitrary constant, but we can take it to be zero by a suitable translation of the surface parallel to the z-axis. Putting $x = f(u)$, $z = g(u)$ and noting that $\cosh^{-1}(u) = \ln(v + \sqrt{v^2 - 1})$, we see that the profile curve in $xz - plane$ has equation

$$z = \sqrt{1 - x^2} - \cosh^{-1}\left(\frac{1}{x}\right). \quad (12)$$

Rotation this curve around the $z - axis$ thus gives a surface which has Gaussian curvature -1 everywhere. Note that, since $u \le 0, x = e^u$ is restricted to the range $0 < x \le 1$.

4. Manifold of Constant Curvature:

Definition (4.1): [9], pp45. By definition, on n-dimensional manifold of constant curvature K is a length space X that is locally isometric to M_k^n. In other words, for every point $x \in X$ there is an $\varepsilon > 0$ and an isometry $\emptyset$ from $B(x, \varepsilon)$ onto a ball $B(\emptyset(x), \varepsilon) \subset M_k^n$.

Theorem (4.2): [9], pp45. Let X be a connected, n-dimensional manifold of constant curvature k. When endowed with the induced length metric, the universal covering of $X$is isometric to M_k^n.

Proof. By definition, a chart $\emptyset : U \rightarrow M_k^n$ is an isometry from an open set $U \subseteq X$onto an open set $\emptyset(U) \subseteq M_k^n$. If$\emptyset': U' \rightarrow M_k^n$ is another chart and if $U \cap U'$ is connected, then there is a unique isometry g∈ Isom(M_k^n) such that $\emptyset$ and $g \circ \emptyset'$ are equal on $U \cap U'$.Consider the set of all pairs $(\emptyset, x)$, where $\emptyset: U \rightarrow M_k^n$ is a chart and $x \in U$. We say that two such pairs $(\emptyset, x)$ and $(\emptyset', x')$ are equivalent if $x = x'$and if the restrictions $\emptyset$ and $\emptyset'$ of to a small neighborhood of x coincide. This is indeed an equivalence relation and the equivalence class of $(\emptyset. x)$ is called the germ of $\emptyset$ at x. Let Xbe the set of all equivalence classes, i.e. the set of all germs of charts. Let $\hat{p}: \hat{X} \rightarrow X$ and $\widehat{D}: \hat{X} \rightarrow M_k^n$be the maps that send the germ of $\emptyset$ at x to x and $\emptyset$ (x) respectively. Notice that there is a natural action of G = Isom(M_k^n) on $\hat{X}$if $\hat{x}$ is the germ of $\emptyset$ at x and if g ∈ G, then g.$\hat{x}$ is the germ of g o $\emptyset$ at x. The map $\widehat{D}: \hat{X} \rightarrow M_k^n$ is G-equivariant: g.($\widehat{D}(\hat{x})$) = $\widehat{D}$(g.$\hat{x}$) if $\hat{x}, \hat{x}' \in \hat{p}^{-1}(x)$, there are a unique $g \in G$ such that g.$\hat{x} = \hat{x}'$. There is a natural topology on$\hat{X}$, called the germ topology, with respect to which p is a covering and D is a local homeomorphism. The basic open sets defining this topology are $U_\emptyset$, where $\emptyset: U \rightarrow M_k^n$ is a chart and $U_\emptyset, \subseteq \hat{X}$is the set of germs of $\emptyset$ at the various points of U. The restriction of $\hat{p}\left(resp. \widehat{D}\right) to U_\emptyset$, is a homeomorphism onto U (resp. $\emptyset(U)$. Moreover, if U is connected then $\hat{p}^{-1}$(U) is the disjoint union of the open sets $U_{g \circ \emptyset}$, where g e G .Thus $\hat{p}: \hat{X} \rightarrow X$ is a covering map (and in particular X is Hausdorff Choose a base point $x_0 \in X$ and a chart $\emptyset$ defined at x_0. Let$\hat{x}_0 \in \hat{X}$ be the germ of $\emptyset$ at x_0, and let $\hat{X}$ be the connected component of $\hat{X}$ containing$\hat{x}_0$. Let p : $\tilde{X} \rightarrow X$ and $D: \tilde{X} \rightarrow M_k^n$ be the restrictions of $\hat{p}$ and $\widehat{D}$to $\tilde{X}$. Let Γ ⊂G b e the subgroup of G that leaves $\tilde{X}$ invariant. Then p : $\tilde{X} \rightarrow X$ is a Galois covering with Galois group Γ and the map D is a local homeomorphism which is Γ-equivariant. If we endow $\tilde{X}$ with the unique length metric $\tilde{d}$ such that p is a local isometry , then D becomes a local isometry. Now we assume that X is complete. Then $(\tilde{X}, \tilde{d})$ is also complete, if D : $\tilde{X} \rightarrow M_k^n$ is a covering. As M_k^n is simply connected and $\tilde{X}$ connected, D must be a homeomorphism. Thus $\tilde{X}$ is simply connected (hence the universal covering of X), and D is an isometry.

5. Manifold of Constant Nonpositive Curvature:

Our major local- global theorem in arbitrary dimensions is the following character- ization of simply – connected manifold of constant no positive curvature.

Theorem(4.1):[3], pp(196-197). If M is a complete connected manifold all of whose sectional curvature are nonpositive , then for any point $p \in M$, $exp_p: T_pM \rightarrow M$ is a covering map. In particular, the universal covering space of M is diffeomorphic to R^n. If M is simply connected , then M itself is diffeomorphic to R^n.

Proof:

The assumption of nonpositive curvature guarantees that p has no conjugate points along any geodesic , which can be shown by using either the conjugate point comparison theorem above (4.1). Therefore, exp_p is a local diffeomorphism on all of T_pM. Let $\tilde{g}$ be the (variable – coefficient) 2-tensor field $exp_p^* \, g$ defined on T_PM. Because exp_p^* is everywhere nonsingular, $\tilde{g}$ is a Riemannian metric and $exp_p: (T_pM, \tilde{g}) \rightarrow (m, g)$ is a local isometry. it then follows from Lemma (4.2) below that $exp_p$is a covering map. The remaining statements of the theorem follow immediately from uniqueness of the universal covering space.

Lemma(4.2):[3], pp197. Suppose $\widetilde{M}$and M are connected Riemannian manifolds with$\widetilde{M}$complete and $\pi: \widetilde{M} \rightarrow$ M is a local isometry. Then M is complete and π is a covering map.

6. Discussion:

After we provided the introduction of manifold of constant negative curvature we explained and discussed that a Riemannian manifold have manifold of constant negative curvature and we know us constant negative curvature which has Gaussian curvature. Also we discussed and proved that the most important local-global theorem of simply-connected manifold of nonpositive sectional curvature is the Catan-Hadamard theorem.

Results:

Manifold of constant negative curvature indicates to know the behavior of some of the functions and manifold and also it revealed the cartan- Hadamard theorem which is considered one of the importance application of the manifold of constant negative curvature.

Conclusion:

Finally, we can say that any manifold of constant negative curvature is Riemannian manifold.

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