ON ROTATION MIXED TYPE SURFACES IN LORENTZ-MINKOWSKI SPACE

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ABSTRACT
In this paper, we study on rotational mixed type surfaces and give two theorems about degeneracy of first fundamental form.

1. INTRODUCTION

It is known that every closed surface in L^3must be of mixed type [1]. The first fundamental form of a mixed type surface is type-changing metric. Type-changing metrics are of interest both mathematics and theoretical physics[2].[3].

Definition 1.1 Let us denote by L^3 the Lorentz-Minkowski 3-space of signature $(+ \quad + \quad -)$. Concider an embedded surface S in L^3. The induced metric of S might be either positive definite, indefinite or degenerate. According to such properties, a point on the surface S is said to be spacelike, timelike or lightlike. If the spacelike, timelike and lightlike point sets are all non-empty, the surface S is said to be *a mixed type surface* [4]

Definition 1.2 A surface is called a *rotation surface* with axis *l* if it is invariant under the action of the group of motions in L^3 which fix each point of the line *l*. [5].

The main result of this paper is as follows:

Theorem A: Let $f: \sum \rightarrow L^3$ be a mixed type rotation surface given by

$$f(u,v) = (u, f(u)cosv, f(u)sinv)$$

with the first fundamental form

$ds^2 = Edu^2 + 2Fdudv + Gdv^2$ and discriminant function $\lambda \coloneqq EG - F^2$

If $f_u \perp f_v$ and $\lambda \coloneqq EG - F^2 = 0$,where $f_u := df(\partial_u)$ and $f_v := df(\partial_v)$ then the surface is either a cylinder or a cone.

Theorem B: Let $f: \sum \rightarrow L^3$ be a mixed type rotation surface given by

$$f(u,v) = (g(u)sinhv, u, g(u)coshv)$$

with the first fundamental form

$ds^2 = Edu^2 + 2Fdudv + Gdv^2$ and discriminant function $\lambda \coloneqq EG - F^2$

If $\lambda \coloneqq EG - F^2 = 0$, then the surface is an hyperboloid of two-sheets.

2. PRELIMINARIES

We denote by L^3 the Lorentz-Minkowski 3-space with the standard Lorentz metric

$$< x, x >_L = x_1^2 + x_2^2 - x_3^2 \; ; \; x = \sum_{i=1}^{3} x_i e_i \in L^3$$

where $S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $e_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

A vector $x \in L^3$ $(x \neq 0)$ is said to be
spacelike if $< x, x >_L > 0$,
timelike if $< x, x >_L < 0$,
lightlike if $< x, x >_L = 0$.

2.1 Mixed type surfaces In this paper, a surface in L^3 is defined to be an embedding $f: \sum \to L^3$ of a connected differentiable 2-manifold $\sum$. A point $p \in \sum$ is said to be a lightlike (resp. spacelike, timelike) point if the image $V_p := df_p\,(T_p\sum)$ of the tangent space $T_p\sum$ is lightlike (resp. spacelike, timelike) 2-subspace of L^3. We denote by LD the set of lightlike points, $\sum_+$ the set of spacelike points and $\sum_-$the set of timelike points. If both $\sum_+ \; and \; \sum_-$ are non-empt, the surface is called a mixed type surface.Denote by ds^2 the first fundamental form of f. ds^2 is the smooth metric on $\sum$ defined by $ds^2 := f^* < , >_L$. Then $p \in \sum$ is a lightlike point if and only if $(ds^2)_p$ degenerate as a symmetric bilinear form on $T_p\sum$. Similarly $p \in \sum$ is a spacelike (resp. timelike) point if and only if $(ds^2)_p$ is positive definite (resp. indefinite). Take a local coordinate neighborhood $(U; u, v)$ of $\sum$. Set

$f_u := df(\partial u)$ $f_v := df(\partial v)$. Then ds^2 is written as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

$$where\ E :=< f_u, f_u >_L, F :=< f_u, f_v >_L, G :=< f_v, f_v >_L$$

If $\lambda := EG - F^2$ then a point $q \in U$ is lightlike (resp. spacelike, timelike) point if and only if $\lambda(q) = 0$ $(resp.\ \lambda(q) > 0,\ \lambda(q) < 0)$ holds. λ is called as *discriminant function.*

2.2 Rotation surfaces In this paper, a rotation surface in L^3 is given by timelike axis as x-axis and spacelike axis as y-axis then the surface is expressed as follows:

$f(u, v) = (u, f(u)cosv, f(u)sinv)$ $\quad if\ the\ axis\ is\ timelike;$
$f(u, v) = (g(u)sinhv, u, g(u)coshv)$ $\quad if\ the\ axis\ is\ spacelike.$

3. PROOF OF MAIN RESULTS

In this section we give the proofs of theorem A and theorem B in the introduction.

3.1 Proof of theorem A Let $f: \sum \to L^3$ be a rotation mixed type surface given by

$$f(u,v) = (u, f(u)cosv, f(u)sinv)$$

Then by direct calculation we obtain

$$EG - F^2 = (1 + f^2 cos2v)(-f^2 cos2v) - f^2 f^2 sin^2 2v$$

$$where\ f = \frac{df}{du}\ and\ f = f(u)$$

If $f_u \perp f_v$ then $f^2 f^2 sin^2 2v = 0$.

We have two cases; $f^2 f^2 = 0\ or\ sin^2 2v = 0$

Case1: If $f^2 f^2 = 0$ then $f = c\ where\ c\ is\ a\ constant.$

Case2: If $sin^2 2v = 0$ then $v = 0, \pi\ or\ \frac{\pi}{2}$

For discriminant function we have

$$EG - F^2 = \begin{cases} -c^2 cos2v & if\quad f^2 f^2 = 0 \\ -f^2(1 + f^2) & if\quad v = 0, \pi \\ f^2(1 - f^2) & if\quad v = \frac{\pi}{2} \end{cases}$$

$$EG - F^2 = 0\ then\ f(u) = c\ or\ f(u) = u + k\ ,\quad k\ any\ constant$$

That means surface is a cylinder or a cone.

3.2 Proof of theorem B Let $f: \sum \to L^3$ be a rotation mixed type surface given by

$$f(u,v) = (g(u)sinhv, u, g(u)coshv)$$

Then by direct calculation we obtain

$$EG - F^2 = g^2(1 - \dot{g}^2)$$

$$where\ \dot{g} = \frac{dg}{du}\ and\ g = g(u)$$

$$EG - F^2 = 0\ then\ g^2 = 0\ or\ (1 - \dot{g}^2) = 0$$

Case1: If $g(u) = 0$ then f does not describe a surface

Case2: If $(1 - \dot{g}^2) = 0$ then $g(u) = \mp u + k\ where\ k\ is\ any\ constant$. That means surface is hyperboloid of two sheets.

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