The 2-Domination Number in Interval-Valued Fuzzy Graphs

Nooraldeen .O. AL-saklady and Mahioub .M. Q. Shubatah

Department of Mathematics, University of Sheba Region, Mareb Yemen.

Department of Mathematics, AL-Baydaa University, AL-Baydaa, Yemen.

E-mail address: $nor771872120@gmail.com$

E-mail address: $Mahioub70@yahoo.com$

Abstract

In this paper we focus on 2- domination number of an interval-valued fuzzy graphs G by using effective edge and is denoted by $\gamma_2(G)$ and obtain some results on $\gamma_2(G)$, the relationship between $\gamma_2(G)$ and some other, known parameters concepts are obtained. Finally the domatic number of interval-valued fuzzy graph is introduced further some results on this concept are investigated.

Keywords: 2-dominating set, 2-dominatic, interval-valued fuzzy graph.

Classification 2010: 03E72,05C69,05C72

1 Introduction

The first idea of domination theory is dated bake to 400 years in India, the theory of dominating in graphs was begun by Ore and Berge [14,4]. Cockayne and Hedetniem studied the concept of domination number in graphs [6]. Zadeh[18]introduced the notion of interval-valued fuzzy set. The domination number of fuzzy graph was introduced by A.Somasundarma and S.Somasundarma using effective edges [16,17]. Some important works in fuzzy graph theory can be found in[2,3,4,9]. Rosenfeld in[15] introduced the nation of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as paths, cycles and connectedness. Nagoorgani and Chandrasekerem in the year (2006) discussed domination in fuzzy graph [12]. The concept of 2-domination in fuzzy graphs was also introduced by Nagoorgani using strong Arcs he considered $\mu(x) = 1, \forall x \in V(G)$ [13]. Hongmei and Lianhua introduced the definition of interval-valued fuzzy graphs[7]. Lou and Yu introduce the concept of cardinality of in intuitionist fuzzy set [10]. Akram and Dudek studied serval properties and operations on interval-valued fuzzy graphs[1]. In this paper we introduce the concept of 2-domination number in interval-valued fuzzy graphs using effective edges. we obtain some interesting results for this Parameter in interval-valued fuzzy graphs.

2 Preliminaries

Definition 2.1 *An interval-valued fuzzy set A on a set V is defined by $A = \{(x), [\mu_A^-(x), \mu_A^+ x] : x \in V\}$, where $\mu_A^-(x)$ and $\mu_A^+(x)$ are fuzzy subsets of V such that $\mu_A^-(x) \leq \mu_A^+(x) \forall x \in V$.*

Definition 2.2 *If $G^* = (\mu^*, \rho^*)$ is a crisp graph, then by an interval-valued* *fuzzy relation* B *on* VV *we maen an interval-valued fuzzy relation on* E *such that* $\mu_B^-(x) \leq \min\{\mu_A^-(x), \mu_A^-(y)\}$ *and* $\mu_B^+(x) \leq \max\{\mu_A^+(x), \mu_A^+(y)\}$ $\forall xy \in E$ *and we write* $B = \{xy, [\mu_B^-(xy), \mu_B^+ xy] : xy \in E\}$.

Definition 2.3 *If* $G^* = (\mu^*, \rho^*)$ *is a crisp graph, then An interval-valued fuzzy graph (simply, IVFG) of a graph* G^* *is a pair* $G = (A,B)$*, where* $A=[\mu^-, \mu^+]$ *is an interval-valued fuzzy set on* V *and* $B= [\mu_B^-, \mu_B^+]$ *is an interval-valued fuzzy relation on* V.

Definition 2.4 *An interval-valued fuzzy graph(IVFG)* $G = (A, B)$ *of a graph* $G^* = (V, E)$ *the order* P *and size* q *are defined to be*
$P = |V| = \sum_{u\in v} \frac{1+\mu_A^+(\nu)-\mu_A^-(\nu)}{2}$ *and* $q = |E| = \sum_{xy\in E} \frac{1+\mu_B^+(xy)-\mu_B^-(xy)}{2}$.

Definition 2.5 *Let* $G = (A, B)$ *be in interval-valued fuzzy graph on* $G^* = (V, E)$ *and* $D \subseteq V$. *Then the cardinality of* D *is defined to be*
$|D| = \sum_{v\in D} \frac{1+\mu_A^+(\nu)-\mu_A^-(\nu)}{2}$.

Definition 2.6 *An IVFG* $G = (V, B)$ *of a crisp graph is called complete fuzzy graph if* $\mu_B^-(x) = \min\{\mu_A^-(x), \mu_A^-(y)\}$ *and* $\mu_B^+(x) = \max\{\mu_A^+(x), \mu_A^+(y)\}$ $\forall xy \in E$, $\forall x, y \in V(G)$ *and is denoted by* K_{μ_A}.

Definition 2.7 *The complement of an IVFG* $G = (A, B)$ *of a graph* $G^* = (V, B)$ *is interval-valued fuzzy graph* $\overline{G} = (\overline{A}, \overline{B})$*, where* $\overline{A} = [\mu_A^-(x), \mu_A^+(x)]$ *and* $\overline{B} = [\mu_B^-(x), \mu_B^+(x)]$ *is defined by* $\overline{\mu_B^+}(xy) = \max\{\mu_A^+(x), \mu_A^+(y)\} - \mu_B^+(xy)$ $\overline{\mu_B^-}(xy) = \max\{\mu_A^-(x), \mu_A(y)\} - \mu_B^-(xy)$, *for all* $xy \in E$.

Definition 2.8 *An interval-valued fuzzy graph* $G = (A, B)$ *of a graph* $G^* = (V, B)$ *is said to be bipartite if the vertex set* V *can be partitioned into two nonempty subsets* V_1 *and* V_2 *such that* $\mu_B^-(xy)) = 0$ *and* $\mu_B^+(xy)) = 0$ *if* $x, y \in V_1$ *or* $x, y \in V_2$ *further if* $\mu_B^-(xy) = \min\{\mu_A^-(x), \mu_A^-(y)\}$ *and* $\mu_B^+(xy) =$

$\max\{\mu_A^+(x), \mu_A^+(y)\}$ $\forall xy \in E$, $\forall x, y \in V(G)$ *for all* $x \in V_1$ *and* $y \in V_2$ *then* G *is called a complete bipartite fuzzy graph and is denoted by* $K_{\mu_A^-,\mu_A^+}$, *where* μ_A^- *and* μ_A^+ *are restriction of* μ_A^- *on* V_1 *and* μ_A^+ *on* V_2.

Definition 2.9 *An edge* $e = uv$ *of an interval-valued fuzzy graph* $G = (A, B)$ *is called an effective edge if*
$\mu_B^-(uv) = \min\{\mu_A^-(u), \mu_A^-(v)\}$ *and* $\mu_B^+(uv) = \max\{\mu_A^+(u), \mu_A^+(v)\}$.

Definition 2.10 *A vertex* v *of an interval-valued fuzzy graph* $G = (A, B)$ *is said that* v *is isolated vertex if* $\mu_B^-(uv) < \min\{\mu_A^-(u), \mu_A^-(v)\}$ *and* $\mu_B^+(uv) < \max\{\mu_A^+(u), \mu_A^+(v)\}$.

3 Main results

In this section, we introduce 2-domination number in interval-valued fuzzy graphs and investigate some properties of γ_2.

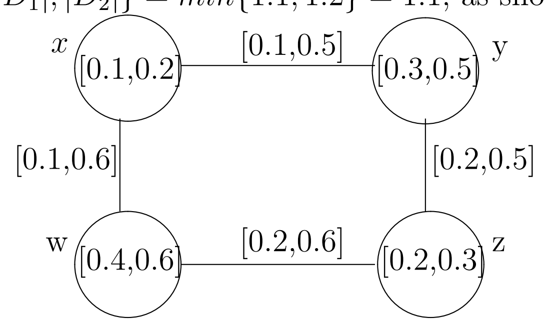
Definition 3.1 Let $G = (A, B)$ be an interval-valued fuzzy graph on V, a subset vertex D of V in IVFG is said to be 2-dominating set in G if for every $v \in V - D$, there is at least two vertices u and $w \in D$ such that $\mu_B^+(uv) = \max\{\mu_A^+(u), \mu_A^+(v)\}$, $\mu_B^-(uv) = \min\{\mu_A^-(u), \mu_A^-(v)\}$, $\mu_B^-(uw) = \min\{\mu_A^-(u), \mu_A^-(w)\}$ and $\mu_B^+(uw) = \max\{\mu_A^+(u), \mu_A^+(w)\}$.

Definition 3.2 A 2-dominating set D of an IVFG $G = (A, B)$ is said to be minimal 2-dominating set of G if $D - \{u\}$ is not 2-dominating set of G for all $u \in D$.

Definition 3.3 The minimum fuzzy cardinality among all 2-dominating set in IVFG $G = (A, B)$ is called the 2-domination number of G and is denoted by $\gamma_2(G)$ or simply γ_2. The maximum fuzzy cardinality taken over all minimal 2-dominating set in G is called the upper 2-domination number of G and is denoted by $\Gamma_2(G)$ or simply Γ_2.

A minimal 2-dominating set D of IVFG $G = (A, B)$ with $|D| = \gamma_2(G)$ dented by γ_2 - set.

Example 3.1 Consider an IVFG G=(A, B) such that V=$\{x, y, w, z\}$ and E=$\{xy, xw, yz, wz\}$. Let A be an interval-valued fuzzy set on V and let B be an interval-valued fuzzy set on $E \subseteq V \times V$ defined by $A = \{x[0.1, 0.2], y[0.3, 0.5], z[0.2, 0.3], w[0.4, 0.6]\}$, $B = \{xw[0.1, 0.6], xy[0.1, 0.5], yz[0.2, 0.5], wz[0.2, 0.6]\}$. In this example, we see that all edges are effective, therefore $D_1 = \{x, z\}$ and $D_2 = \{y, w\}$ are minimal 2-dominating set of IVFG G, since $|D| = \sum_{v \in D} \frac{1+\mu_A^+(v)-\mu_A^-(v)}{2}$, then $|D_1| = \frac{1+0.2-0.1}{2} + \frac{1+0.3-0.2}{2} = 1.1$ and $|D_2| = \frac{1+0.5-0.3}{2} + \frac{1+0.6-0.4}{2} = 1.2$, thus $\gamma_2(G) = min\{|D_1|, |D_2|\} = min\{1.1, 1.2\} = 1.1$, as shown in Figure 3.1.



$Fig.$ 3.1

Theorem 3.1 Every 2-dominating set of interval-valued fuzzy graphs $G = (A, B)$ is a dominating set of G.

Theorem 3.2 Let $G = (A, B)$ be any IVFG and $\nu \in V(G)$, if ν has only one neighbour, Then ν belong to every 2-dominating set of G.

Proof : Let G be an IVFG and $\nu \in V(G)$ has at most one neighbour in G. Let D be 2-dominating set in G .

Cace(i): suppose that ν has no neighbours in G. (i.e $N_\nu = \phi$).
Then any vertex in D; $|N_\nu| = 0$ (i.e ν is an isolated vertex dose not dominates ν).

Cace(ii): suppose that ν has only one neighbour in G and suppose that $\nu \notin D$. Then $\nu \in V - D$.

Since D is 2- dominating set in G, then there are at least two vertices in D, which dominate ν. (i.e ν has two neighbours u and w) but ν has only one neighbor. Thus D is not 2- dominating set. Which is contradiction to assumption. Hence $\nu \in D$.

Corollary 3.1 if ν is an end vertex in IVFG G. Then ν must be in every 2- dominating set.

Theorem 3.3 Let $G = (A, B)$ be any IVFG and D be minimal 2-dominating set of G if $G \neq K_P$ and $G \neq K_{n,m}$.

Then $V - D$ need not be a 2-dominating set of G.

Proof: Let D be 2-dominating set and let $\nu \in V(G)$, Suppose that ν has only one neighbour in G. Then ν dominated only by one vertex in $V - D$. Therefore ν must be in every minimal 2-dominating set of G. Then $V - D$ has either one neighbour of ν or has no any neighbour of ν. Thus $V - D$ is not 2-dominating set of G .

Now, suppose that every vertex in D has at least two neighbours in $V - D$ in this case every vertex in D is dominated by at least two vertices in $V - D$. Thus $V - D$ is 2-dominating set of G. Therefore $\nu \in V - D$ and thus $D - \{\nu\}$ is not minimal 2-dominating set which is contradiction. Hence $V - D$ is not 2-dominating set.

To show that the conditions $G \neq K_P$ and $G \neq K_{n,m}$, in the above theorem is important, we give the following examples.

Example 3.2 Let G is be a complete interval-valued fuzzy graph which given in the Figure 3.2, where A=$\{x$[0.1,0.2],y[0.3,0.5],z[0.2,0.3],w[0.4,0.6]$\}$, and B=$\{xy$[0.1,0.5],xw[0.1,0.6],xz[0,1,0.3],yw[0.3,0.6],yz[0.2,0.5],wz0.2,0.6]$\}$. we observe that $D_1 = \{x, z\}$ and $D_2 = \{y, w\}$, are minimal 2-dominating set. Hence D_1 is 2-dominating set of G, therefore $V - D_1$ also is 2-dominating set of G.

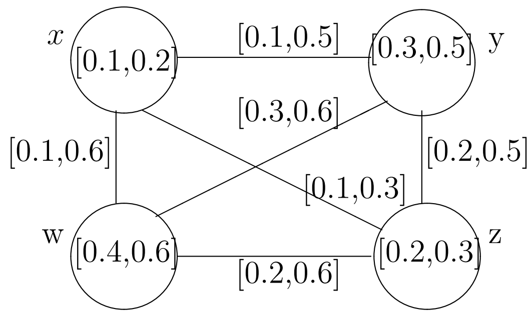


Fig. 3.2

Example 3.3 let $G = K_{\mu_1,\mu_2}$ is a complete bipartite an interval-valued fuzzy graph and given in Figure 3.3, $\mu_B^-(vu) = \min\{\mu_A^-(v), \mu_A^-(u)\}$ and $\mu_B^+(vu) = \max\{\mu_A^+(v), \mu_A^+(u)\}$, where,
$A = \{v_1[0.1, 0.2], v_2[0.2, 0.3], v_3[0.3, 0.4], v_4[0.4, 0.5], v_5[0.5, 0.6]\}$.
We observe that $D_1 = \{v_4, v_5\}$ and $D_2 = \{v_1, v_2, v_3\}$ are minimal 2-dominating set of $G = K_{\mu_1,\mu_2}$, but we see that $V - D_1$ is also 2-dominating set of $G = K_{\mu_1,\mu_2}$.

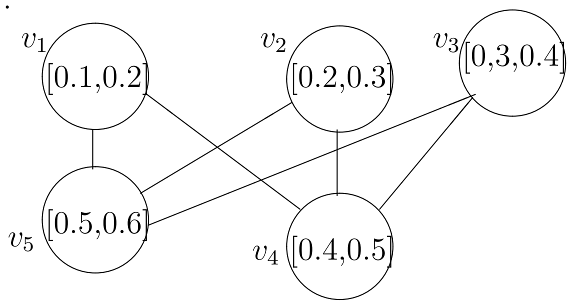


Fig. 3.3

Example 3.4 Let G be an interval-valued fuzzy graph given in Figure 3.4, where, $A = \{v_1[0.1, 0.2], v_2[0.2, 0.3], v_3[0.3, 0.4], v_4[0.4, 0.5]\}$, $\mu_B^-(vu) = \min\{\mu_A^-(v), \mu_A^-(u)\}$ and $\mu_B^+(vu) = \max\{\mu_A^+(v), \mu_A^+(u)\}, \forall u, v \in V$.
We see that $D = \{v_1, v_3, v_4\}$, but $V - D$ is not 2-dominating set of G.

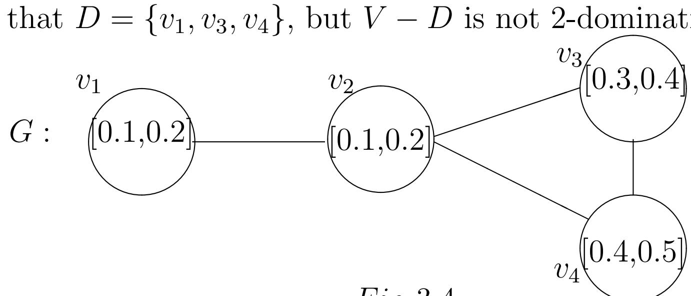


Fig 3.4

In the following results we give $\gamma_2(G)$ for some standard interval-valued fuzzy graph.

Theorem 3.4 Let G be a star interval-valued fuzzy graph then $\gamma_2(G) = P - |u|$ such that $d_N(u) = \Delta_N(G)$, (u is the root).

Proof:Let (G) be a star interval-valued fuzzy graph and $u \in V(G)$ be a root of G when $d_N(u) = \Delta_N(G)$ then all vertices in graph G have only one neighbor except the vertex u. Thus $V - \{u\}$ is 2-domination set of G and hence, $\gamma_2(G) = |V - \{u\}| = P - |u|$.

The following theorem give γ_2 for K_{μ_1,μ_2}.

Theorem 3.5 Let $G = K_{\mu_1,\mu_2}$ is a complete bipartite fuzzy graph then $\gamma_2(G) = \min\{|V_1|, |V_2|\}$

Proof: Let $(G) = K_{\mu_1,\mu_2}$be a complete bipartite interval-valued fuzzy graph. Then $\mu_B^-(uv)) = 0$ and $\mu_B^+(uv)) = 0$ if $u, v \in V_1$ or $u, v \in V_2$ further if $\mu_B^-(uv) = \min\{\mu_A^-(u), \mu_A^-(v)\}$ and $\mu_B^+(uv) = \max\{\mu_A^+(u), \mu_A^+(v)\}$ $\forall uv \in E$, $\forall$u$, v \in V(G)$ for all $u \in V_1$ and $v \in V_2$. We say that every vertex in V_1 dominates all vertices in V_2 and the vice versa. Then V_1 and V_2 are 2-dominating sets of G. Therefor $V_1 and\ V_2$ are minimal 2-dominating sets of G. Hence $\gamma_2(G) = \min\{|V_1|, |V_2|\}$, where $|V_1| = \sum_{v \in V_1} \frac{1+\mu_A^+(v)-\mu_A^-(v)}{2}$. And$|V_2| = \sum_{u \in V_2} \frac{1+\mu_A^+(u)-\mu_A^-(u)}{2}$.

Example 3.5 In example 3.3 we have seen that the vertex subsets, $D_1 = \{v_4, v_5\}$ and $D_2 = \{v_1, v_2, v_3\}$ are minimal 2-dominating sets in G. Since $|D_1| = \sum_{v \in D_1} \frac{1+\mu_A^+(v)-\mu_A^-(v)}{2}$, then $|D_1| = \sum_{v_4 \in D_1} \frac{1+0.5-0.4}{2} + \sum_{v_5 \in D_1} \frac{1+0.6-0.5}{2} = 1.1$ and $|D_2| = \sum_{v_1 \in D_2} \frac{1+0.2-0.1}{2} + \sum_{v_2 \in D_2} \frac{1+0.3-0.2}{2} + \sum_{v_3 \in D_2} \frac{1+0.4-0.3}{2} = 1.65$. Therefore $\gamma_2(K_{\mu_1,\mu_2}) = min\{|D_1|, |D_2|\} = min\{1.1, 1.65\} = 1.1$.

The following theorem gives γ_2 of a complete interval-valued fuzzy graph.

Theorem 3.6 If $G = K_\mu$ is a complete interval-valued fuzzy graph, then

$\gamma_2(K_\mu) = \min\{|u| + |v|\}$, $u, v \in V(G)$

Proof: LetK_μ is a complete interval-valued fuzzy graph, then every vertex in K_μ is a neighbour of all other vertices in G. Therefore any set of two vertices in K_μ will be a 2-dominating set of K_μ. Thus we conclude.

Example 3.6 In example 3.2 if G is a complete interval-valued fuzzy graph, we have seen that the vertex subsets, $D_1 = \{x, y\}$, $D_2 = \{x, z\}$, $D_3 = \{x, w\}$, $D_4 = \{y, z\}$, $D_5 = \{y, w\}$ and $D_6 = \{z, w\}$ are minimal 2-dominating sets of G. Since $|D_1| = \sum_{v \in D_1} \frac{1+\mu_A^+(v)-\mu_A^-(v)}{2}$, then $|D_1| = \sum_{x \in D_1} \frac{1+0.2-0.1}{2} + \sum_{y \in D_1} \frac{1+0.2-0.1}{2} = 1.15$, $|D_2| = 1.1$, $|D_3| = 1.15$, $|D_4| = 1.15$, $|D_5| = 1.2$ and $|D_6| = 1.15$. Hence $\gamma_2(K_\mu) = \min\{|D_1|, |D_2|, |D_3|, |D_4|, |D_5|, |D_6|\}$. Then $\gamma_2(K_\mu) = |D_2| = 1.1$.

In the above example we can see that $V - D$ is also 2-dominating set in G.

Theorem 3.7 For any interval-valued fuzzy graph, if every vertex of G has unique neighbor then $\gamma_2(G) = p$.

Proof: Let G be any interval-valued fuzzy graph, and all vertices of G has only one neighbor. Then V is only 2-dominating set of G. Hence we conclude.

Theorem 3.8 For any interval-valued fuzzy graph, $\gamma_2(G) + \gamma(G) \leq 2p$ and equality hold if $\mu_B^-(uv) < \min\{\mu_A^-(u), \mu_A^-(v)\}$ and $\mu_B^+(uv) < \max\{\mu_A^+(u), \mu_A^+(v)\}$, $\forall u, v \in V(G)$.

Theorem 3.9 Let $G = (A, B)$ be any interval-valued fuzzy graph, if $G^* = nK_2$, then $\gamma_2(G) = np$.

Proof: Let G any IVFG, such that $G^* = nK_2$, since K_2 is a complete interval-valued fuzzy graph with two vertices. Then every vertex in K_2 is an end vertex. Thus every vertex in K_2 is in every 2-dominating set of G. Therefore every vertex in an interval-valued fuzzy graph G is in every minimal 2-dominating set of G. Hence $\gamma_2(G) = np$.

Theorem 3.10 For any IVFG $G = (A, B)$, $\gamma_2(G) + \gamma_2(\overline{G}) \leq 2p$, farther equality hold if $0 < \mu_B^-(uv) < \min\{\mu_A^-(u), \mu_A^-(v)\}$ $\forall uv \in B$, $\forall$u, $v \in A$ and $0 < \mu_B^+(uv) < \max\{\mu_A^+(u), \mu_A^+(v)\}$ $\forall uv \in B$, $\forall$u, $v \in A$

Proof: The inequality is trivial since $\gamma_2(G) \leq p$ and $\gamma_2(\overline{G}) \leq p$.

Nwo, since $0 < \mu_B^-(uv) < \min\{\mu_A^-(u), \mu_A^-(v)\}$ $\forall uv \in B$, $\forall$u, $v \in A$ and $0 < \mu_B^+(uv) < \max\{\mu_A^+(u), \mu_A^+(v)\}$ $\forall uv \in B$, $\forall$u, $v \in A$. Then $0 < \overline{\mu_B(uv)} = 1 - \mu_B(uv)$. Thus $\gamma_2(G) = p$ and $\gamma_2(\overline{G}) = p$. Hence $\gamma_2(G) + \gamma_2(\overline{G}) = p + p = 2p$.

Corollary 3.2 Let G be an IVFG such that both G and $\overline{G}$ have no isolated vertices. $\gamma_2(G) + \gamma_2(\overline{G}) \leq p$. Farther equality holds if $\gamma_2(G) = \gamma_2(\overline{G}) = \frac{p}{2}$.

Theorem 3.11 For any interval-valued fuzzy graph $G = (A, B)$ if $G \neq K_P$, $\gamma_2(G) \leq p - \delta_N$.

Proof: Suppose that G be any IVFG and $G \neq K_p$. Let $v \in V(G)$ such that $d_N(v) = \delta_N(G)$ and let D be minimal 2- dominating set of G. Then $V - N(v)$ is 2-dominating set of G. Hence, $\gamma_2(G) \leq |V - N(v)| = p - \delta_N(G)$.

Corollary 3.3 For any an IVFG $G = (A, B)$ such that $G \neq K_p$, $\gamma_2(G) \leq p - \delta_E(G)$.

Proof: Since $\Delta_E \leq \Delta_N$ and $\delta_E \leq \delta_N$. Then $p - \delta_N \leq p - \delta_E$. Hence by theorem 3.12 $\gamma_2(G) \leq p - \delta_E$.

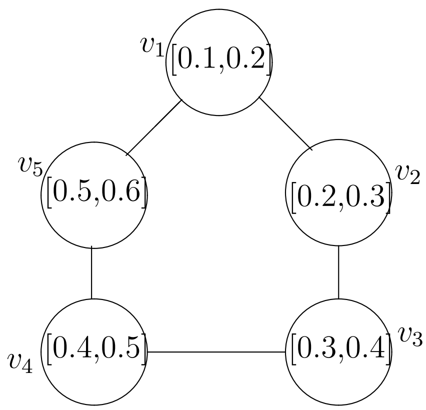
Corollary 3.4 If $G = K_{n,m}$ complete bipartite interval-valued fuzzy graph, then $\gamma_2(G) = p - \delta_N$.

Definition 3.4 A partition $p = \{D_1, D_2, ..., D_m\}$ of $V(G)$ is 2-domatic partition of an interval-valued fuzzy graph G if D_i is 2-dominating sets of G $\forall i$.

A fuzzy cardinality of a partition P is denoted by $||p||$ and is defined as $||p|| = \sum_{i=1}^{m} \frac{\mu(D_i)}{|D_i|}$, where $|D_i|$ is number of vertices in D_i, the 2-domatic number of interval-valued fuzzy graph G is the maximum fuzzy cardinality of $||P||$. That is, $d_2(G) = max\{||p_i||\} = max\{\sum_{i=1}^{m} \frac{\mu(D_i)}{|D_i|}\}$.

Example 3.7 Let $G = (A, B)$ be an interval-valued fuzzy graph, which

given in Figure 3.7 such that $A = \{v_1, v_2, v_3, v_4, v_5\}$, and every edges are effective. There are the following partitions of $V(G)$ into 2-dominating set .
$p_1 = \{v_1, v_3, v_5\}, \{v_2, v_4, v_5\}$,
$p_2 = \{v_2, v_4, v_5\}, \{v_1, v_3, v_5\}$,
$p_3 = \{v_1, v_3, v_4\}, \{v_2, v_4, v_5\}$,
$p_4 = \{v_1, v_2, v_3, v_5\}$ with,
$||p_1|| = \frac{\frac{1+0.2-0.1}{2}+\frac{1+0.4-0.3}{2}+\frac{1+0.6-0.5}{2}+\frac{1+0.3-0.2}{2}+\frac{1+0.5-0.4}{2}+\frac{1+0.6-0.5}{2}}{3} = 1.1$
$||p_2|| = \frac{3.3}{3} = 1.1$
$||p_3|| = \frac{3.3}{3} = 1.1$
$||p_4|| = \frac{2.2}{4} = 0.55$
Hence $d_2(G) = 1.1$.



$Fig\ 3.7$

Theorem 3.12[11] Every dominating set of a fuzzy graph $G = (V, \mu, \rho)$ is a dominating set of a crisp graph G^* but the converse is not true.

Theorem 3.13[11] Let $G = (V, \mu, \rho)$ be *fuzzy graph*. A dominating set D of $G^* = (\mu^*, \rho^*)$ is a dominating set of G if $\rho(u, v) = \mu(u) \wedge \mu(v)\ \forall u, v \in V(G)$.
Consequently with the above results we get the following theorem gives the relationship between 2-dominating sets of an interval-valued fuzzy graphs $G = (A, B)$ and crisp graph G^*.

Theorem 3.14 Every 2-dominating set D of an interval-valued fuzzy graph $G = (A, B))$ is a 2-dominating set of G^* but the converse is not true.

Proof: Let $G = (A, B)$ an interval-valued fuzzy graph and D is 2-dominating set of G and $G^* = (V, E)$, then for each $v \in V - D$, v has at least two neighbours u, w in D, such that $0 < \mu_B^-(uv) = \min\{\mu_A^-(u), \mu_A^-(v)\}$ $\forall uv \in E$, $\forall$u, $v \in V$ and $0 < \mu_B^+(uv) = \max\{\mu_A^+(u), \mu_A^+(v)\}$ $\forall uv \in E$, $\forall$u, $v \in V$. And $0 < \mu_B^-(vw) = \min\{\mu_A^-(v), \mu_A^-(w)\}$ $\forall vw \in E$, $\forall$v,w$\in V$ and $0 < \mu_B^+(vw) = \max\{\mu_A^+(v), \mu_A^+(w)\}$ $\forall vw \in E$, $\forall$v, $w \in V$. Therefor v has two neighbours in D. Hence the theorem.

The following example we show that the converse of the above theorem is not true.

Example 3.8 Let $G = (A, B)$ be an interval-valued fuzzy graph given in Figure3.8 and $G^* = (V, E)$ given in Figure 3.9

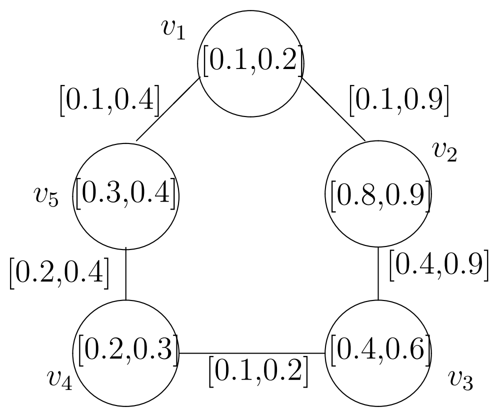


Fig 3.8

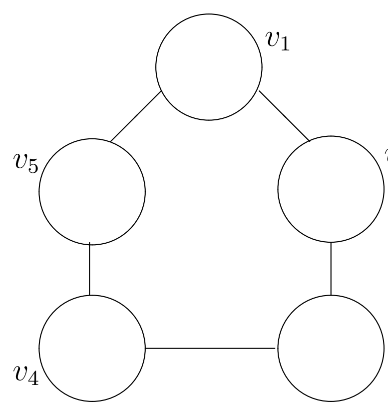


Fig. 3.9

We see that $D = \{v_1, v_3, v_5\}$ is 2-dominating set of G^* but is not 2-dominating set of G.

Theorem 3.15 Let $G = (A, B)$ be an interval-valued fuzzy graph,

A 2-dominating set D of $G^* = (\mu^*, \rho^*)$ is 2-dominating set of an interval-valued fuzzy graph, G if every edges of G is effective.

Proof: Let D be $\gamma_2 - set$ of $G^* = (\mu^*, \rho^*)$, then $\forall v \in V - D$, there are two vertices u and $w \in D$ such that $(u, v) \in \rho^*$ and $(v, w) \in \rho^*$.

Therefor $\mu_B(uv)) > 0$ and $\mu_B(vw) > 0$ and since

$0 < \mu_B^-(uv) = \min\{\mu_A^-(u), \mu_A^-(v)\}$ $\forall uv \in E$, $\forall u, v \in V$ and
$0 < \mu_B^+(uv) = \max\{\mu_A^+(u), \mu_A^+(v)\}$ $\forall uv \in E$, $\forall u, v \in V$.
And $0 < \mu_B^-(vw) = \min\{\mu_A^-(v), \mu_A^-(w)\}$ $\forall vw \in E$,$\forall$v$, w \in V$ and
$0 < \mu_B^+(vw) = \max\{\mu_A^+(v), \mu_A^+(w)\}$ $\forall vw \in E$, $\forall$v$, w \in V$.
Thus v has two neighbours in D. Hence D is 2-dominating set of G.

A consequence of the above theorem we have the following result.

Theorem 3.16 Let $G = (A, B)$ be an interval-valued fuzzy graph, then $\gamma_2(G) \leq \gamma_2(G^*)$.
Furthermore equality hold if $\mu_B(x, y) = [1, 1] = [\mu_B^-, \mu_B^+]\forall(x, y) \in \rho^*$.

Proof: Let D be 2-dominating set of an interval-valued fuzzy graph G, then by theorem 3.15 D is 2-dominating of a crisp graph $G^* = (\mu^*, \rho^*)$. Hence $\gamma_2(G) \leq \gamma_2(G^*)$. If D is 2-dominating set of G^* and $\mu_B(x, y) = [1, 1]$, then by theorem 3.16 D is 2-dominating set of an interval-valued fuzzy graphs G. Hence $\gamma_2(G) = \gamma_2(G^*)$.

Definition 3.5[8] A *subset* $S \subseteq V(G)$ *is* a *double dominating set of* G if S *dominates every vertex* of G at least *twice*.

The double domination number of G is the minimum fuzzy cardinality of double dominating sets of G and is denoted by $\gamma_{dd}(G)$ or simply γ_{dd}.

Theorem 3.17 Every double dominating set D of an interval-valued fuzzy graph G is a 2-dominating set of G.

Proof: Let $G = (A, B)$ be any interval-valued fuzzy graph and D is double dominating set of G. Then $\forall v \in A$ there exists at least tow vertices u and w in D such that $\mu_B^-(uv) = \min\{\mu_A^-(u), \mu_A^-(v)\}$ $\forall uv \in B$, $\forall u, v \in A$ and
$\mu_B^+(uv) = \max\{\mu_A^+(u), \mu_A^+(v)\}$ $\forall uv \in B$, $\forall u, v \in A$.
And $\mu_B^-(vw) = \min\{\mu_A^-(v), \mu_A^-(w)\}$ $\forall vw \in B$,$\forall$v$, w \in A$ and
$\mu_B^+(vw) = \max\{\mu_A^+(v), \mu_A^+(w)\}$ $\forall vw \in B$, $\forall$v$, w \in A$. Then $\forall v \in V - D$ there exist at least tow vertices u and w in D such that u and w dominate v. Thus

D is 2-dominating set of G.

Corollary 3.6 For any interval-valued fuzzy graph G, $\gamma_2(G) \leq \gamma_{dd}(G)$.

Theorem 3.18[5] For any graph G, $\lceil \frac{p}{\Delta_E+1} \rceil \leq \gamma(G) \leq n - \Delta_E(G)$.

Theorem 3.19 For any interval-valued fuzzy graph $\gamma_2(G) \geq \frac{p}{\Delta_E+1}$.

Proof: Since $\lceil \frac{p}{\Delta_E+1} \rceil \leq \gamma(G)$ by theorem 3.18 and since $\gamma_2(G) \geq \gamma(G)$. Then $\gamma_2(G) \geq \gamma(G) \geq \frac{p}{\Delta_E+1} \Longrightarrow \gamma_2(G) \geq \frac{p}{\Delta+1}$.

Corollary 3.6 Let G an interval-valued fuzzy graph, then

$\gamma_2(G) \leq \gamma(G) + \beta_\circ(G)$

Corollary 3.7 Let G an interval-valued fuzzy graph, then

$\gamma_2(G) \leq 2\beta_\circ$.

Theorem 3.20 Let G an interval-valued fuzzy graph, then

$\gamma_2(G) \leq \frac{\gamma(G)+3\beta_\circ(G)}{2}$.

Proof: let G be any interval-valued fuzzy graph, since $\gamma_2(G) \leq \gamma(G) + \beta_\circ(G)$ and $\gamma_2(G) \leq 2\beta_\circ(G)$ by Corollary 3.8.

Then $2\gamma_2(G) \leq \gamma(G) + 3\beta_\circ(G)$. Thus we conclude.

Theorem 3.21 Let G an interval-valued fuzzy graph and G is a cycle with n vertex, then the 2-domination number of G given by:

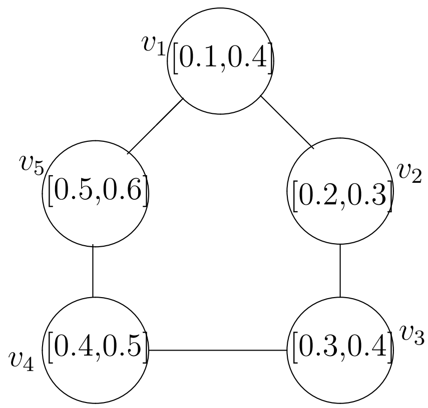
$\gamma_2(G) = min\{|D_1|, |D_2|\}$, such that$|D_1| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mu(v_{1+2i})$ and

$|D_2| = (\sum_{i=1}^{\frac{n}{2}} \mu(v_{2i})) \cup min\{v_1, v_n\}$.

Example 3.9 Let $G = (A, B)$ be an interval-valued fuzzy graph and G is a cycle, given in Figure 3.10. We observe that D_1 and D_2 are minimal 2-domination set of IVFG such that $D_1 = \{v_1, v_3, v_5\}$ and $D_2 = \{v_2, v_4\} \cup min\{v_1, v_5\}$. Then, $|D_1| = \frac{1+0.4-0.1}{2} + \frac{1+0.4-0.3}{2} + \frac{1+0.6-0.5}{2} = \frac{3.5}{2} = 1.75$.

And $|D_2| = \frac{1+0.3-0.2}{2} + \frac{1+0.5-0.4}{2} + min\{\frac{1+0.4-0.1}{2}, \frac{1+0.6-0.5}{2}\} = 0.55 + 0.55 + \min\{0.65, 0.55\} = 1.65$

$\Longrightarrow \gamma_2(G) = min\{|D_1|, |D_2|\} = \min\{1.75, 1.65\}$=1.65.

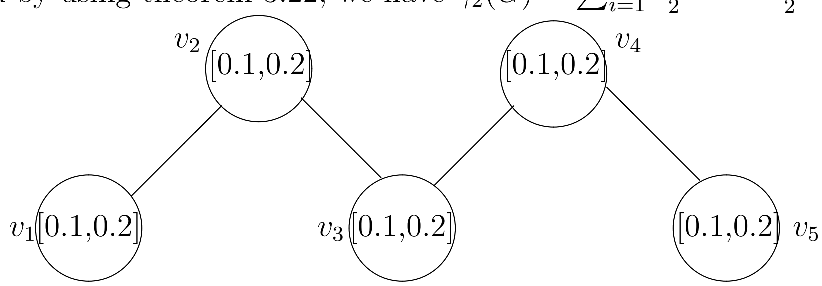


$Fig.$ 3.10

Theorem 3.22 If $G = P_n$ is an interval-valued fuzzy path with n vertices and $\mu(v) = t$, $|v| = t$, $\forall v \in V(P_n)$ then,

$$\gamma_2(G) = \begin{cases} \sum_{i=1}^{n} \frac{p+t}{2} & \text{if } n \text{ odd;} \\ \sum_{i=1}^{n} \frac{p}{2} + t & if\ n\ even; \end{cases}$$

Example 3.10 $G = (A, B)$ be an interval-valued fuzzy graph and G is a path, which given in Figure 3.11. We observe that $D = \{v_1, v_3, v_5\}$ is minimal 2-dominating set with $\gamma_2(G) = |D| = \frac{1+0.2-0.1}{2} + \frac{1+0.2-0.1}{2} + \frac{1+0.2-0.1}{2} = 1.65$. and by using theorem 3.22, we have $\gamma_2(G) = \sum_{i=1}^{n} \frac{p+t}{2} = \frac{2.75+0.55}{2} = 1.65$.

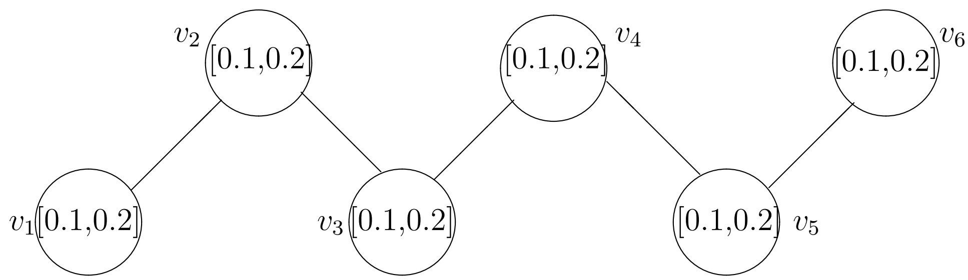


$Fig.$ 3.11

And in Figure 3.12 we observe that $D = \{v_1, v_3, v_5, v_6\}$ is minimal 2-dominating set with,

$\gamma_2(G) = |D| = \frac{1+0.2-0.1}{2} + \frac{1+0.2-0.1}{2} + \frac{1+0.2-0.1}{2} + \frac{1+0.2-0.1}{2} = 2.2$,

and by using theorem 3.22, we have $\gamma_2(G) = \sum_{i=1}^{n} \frac{p}{2} + t = \frac{3.3}{2} + 0.55 = 2.2$.



$Fig.$ 3.12

References

[1] M. Akram and W. A. Dudek, "Interval-valued fuzzy graphs", Comput. Math. Appl. 61, (2011), 289-299.

[2] P. Bhattacharya, "Some remarks on fuzzy graphs", Pattern Recognit. Lett. 6, (1987), 297-302.

[3] K. R. Bhutani and A. Battou, "On M-strong fuzzy graphs", Inform. Sci. 155, (2003), 103-109.

[4] C. Berge, "Graphs and Hyper Graphs", North- Holland, Amsterdam, 309, (1973).

[5] C.Berge, "Theory of graphs and it's Applications "Methuen ,London,1962.

[6] E. J. Cockayne and S. Hedetniemi, "Towards a theory of domination in graphs, Networks", 7, (1977), 247-261.

[7] J. Hongmei and W. Lianhua, "Interval-valued fuzzy subsemigroups and subgroups associated by interval-valued fuzzy graphs", WRI Global Congress on Intelligent Systems (2009), 484-487.

[8] J. Harant and M. A. Henning "On double domination in graphs" Graph Theory 25, (2005), 29-34.

[9] F. Hararay, "Graph Theory", 3rd Edition, Addison-Wesley, Reading, MA, 1972.

[10] Y. Lou and C. Yu, A fuzzy optimization method for multi-criteria decision making problem based on the inclusion degrees of intuitionistic fuzzy sets, J. Inf. Computing Sci. 3(2), (2008), 146-152.

[11] Q.M. Mahioub and N.D. Soner, "Fuzzy covering and fuzzy neighborhood set in fuzzy graphs", Journal of Ultra Scientist of Physical Sciences, 19(3), (2007).

[12] A. Nagoorgani and V. T. Chandrasekaran, "Domination in fuzzy graph", Advances in Fuzzy Sets and Systems, 1 (1), (2006), 17-26.

[13] A. Nagoorgani and K. Prasanna Devi"2-Domination in fuzzy graphs", 9(1), (2015), 119-124.

[14] O. Ore, "Theory of graphs", Amer. Math. Soc. Colloq. Publi., Amer. Math. Soc, Providence, RI 38, (1962).

[15] A. Rosenfeld, "Fuzzy graphs, Fuzzy Sets and their Applications to Cognitive and Decision Processes" (Proc. U.S.-Japan Sem., Univ. Calif., Berkeley, Calif.,(1974).

[16] A. Somasundaram and S. Somasundaram, "Domination in fuzzy graphs-I", Pattern Recognition Letters, 19, (1998), 787-791.

[17] A. Somasundaram, "Domination in fuzzy graphs-II", Journal of Fuzzy Mathematics, 20, (2004), 281-288.

[18] L. A. Zadeh, "The concept of a linguistic and application to approximate reasoning I", Inform. Sci. 8, (1975), 149-249.