Some applications of oscillation criteria for fractional differential and difference equations

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Abstract: In this paper, we present some applications of oscillation criteria for fractional differential and difference equations. The presented examples consist of both continuous and discrete analysis as special cases.

Key–Words: oscillation; fractional differential equations; fractional difference equations

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1 Introduction

Oscillation belongs to the range of qualitative properties analysis. In the last few decades, research for oscillation of various equations including differential equations, difference equations has been a hot topic in the literature, and much effort has been done to establish new oscillatory criteria for these equations so far (for example, see [1-12], and the references therein).

In this paper, we present some applications for oscillation of some fractional differential and difference equations with damping term, where the fractional derivative is defined as the conformable fractional derivative [13]. These fractional differential and difference equations are special cases of the following conformable fractional dynamic equation with damping term on time scales:

$$(a(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)})^{(\alpha)}+p(t)[r(t)x^{(\alpha)}(t)]^{(\alpha)}+q(t)x(t)=0,\ t\in\mathbb{T}_0, \tag{1.1}$$

where $\alpha\in(0,1]$, $\mathbb{T}$ is an arbitrary time scale, $\mathbb{T}_0=[t_0,\infty)\bigcap\mathbb{T}$, $t_0>0$, $a,\ r,\ p,\ q\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$. For the sake of convenience, denote $\delta_1(t,t_i)=\int_{t_i}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s$, where $\widetilde{p}(t)=t^{\alpha-1}p(t)$.

2 Applications

Based on the obtained results in [14, 15], we will present some applications for the established results above.

2.1 Fractional differential equation

First we consider the following fractional differential equation with damping term:

$$\{\sqrt{t}[t^{-\frac{1}{2}}x^{(\frac{1}{2})}(t)]^{(\frac{1}{2})}\}^{(\frac{1}{2})}+t^{-\frac{5}{2}}[t^{-\frac{1}{2}}x^{(\frac{1}{2})}(t)]^{(\frac{1}{2})}+t^{-\frac{3}{2}}x(t)=0,\ t\in[2,\infty). \tag{2.1}$$

Related to (1.1), one has $\mathbb{T}=\mathbb{R}$, $\alpha=\frac{1}{2}$, $a(t)=\sqrt{t}$, $p(t)=t^{-\frac{5}{2}}$, $q(t)=t^{-\frac{3}{2}}$, $\widetilde{p}(t)=t^{-\frac{1}{2}}p(t)=t^{-3}$, $r(t)=t^{-\frac{1}{2}}$, $t_0=2$. So $\mu(t)=\sigma(t)-t=0$, which means $-\frac{\widetilde{p}}{a}\in\mathfrak{R}_+$. Then $e_{-\frac{\widetilde{p}}{a}}(t,t_0)=e_{-\frac{\widetilde{p}}{a}}(t,2)=\exp(-\int_2^t\frac{\widetilde{p}(s)}{a(s)}ds)$. Moreover,

$$1 > \exp(-\int_2^t \frac{\widetilde{p}(s)}{a(s)}ds) \geq 1 - \int_2^t \frac{\widetilde{p}(s)}{a(s)}ds = 1 - \int_2^t s^{-\frac{7}{2}}ds = 1 + \frac{2}{5}[t^{-\frac{5}{2}} - 2^{-\frac{5}{2}}] > \frac{3}{5}.$$

So one can deduce that

$$\int_{t_0}^{\infty} \frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s = \int_{t_0}^{\infty} \frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}\Delta s = \int_{t_0}^{\infty} \frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}ds$$
$$> \frac{3}{5}\int_2^{\infty} \frac{1}{\sqrt{s}}s^{\alpha-1}ds = \frac{3}{5}\int_2^{\infty} \frac{1}{s}ds = \infty,$$

and

$$\int_{t_0}^{\infty} \frac{1}{r(s)}\Delta^\alpha s = \int_{t_0}^{\infty} \frac{1}{r(s)}s^{-\frac{1}{2}}ds = \int_{t_0}^{\infty} 1ds = \infty.$$

Furthermore, one has

$$\int_{t_0}^{\infty}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s)\Delta^\alpha\tau]\Delta^\alpha\xi$$
$$= \int_{t_0}^{\infty}\xi^{\alpha-1}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau]\Delta\xi$$
$$= \int_{t_0}^{\infty}\xi^{\alpha-1}[\frac{1}{r(\xi)}\int_{\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_{\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}ds)d\tau]d\xi$$
$$= \int_2^{\infty}[\int_{\xi}^{\infty}\tau^{-\frac{1}{2}}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,2)}{\sqrt{\tau}}\int_{\tau}^{\infty}\frac{1}{s^2 e_{-\frac{\widetilde{p}}{a}}(s,2)}ds)d\tau]d\xi$$
$$> \frac{3}{5}\int_2^{\infty}[\int_{\xi}^{\infty}(\frac{1}{\tau}\int_{\tau}^{\infty}\frac{1}{s^2}ds)d\tau]d\xi = \frac{3}{5}\int_2^{\infty}[\int_{\xi}^{\infty}\frac{1}{\tau^2}d\tau]d\xi = \frac{3}{5}\int_2^{\infty}\frac{1}{\xi}d\xi = \infty.$$

On the other hand, for a sufficiently large t_2, we have

$$\delta_1(t,t_2) = \int_{t_2}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s = \int_{t_2}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}\Delta s$$
$$= \int_{t_2}^{t}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}ds > \frac{3}{5}\int_{t_2}^{t}\frac{1}{s}ds \to \infty\ (t\to\infty).$$

So there exists a sufficiently large $t_3 > t_2$ such that $\delta_1(t,t_2) > 1$ for $t \in [t_3,\infty)$.

Setting $\phi(t) = t,\ \varphi(t) = 0$ in [14, (2.9)], one can obtain that

$$\int_{t_3}^{t}[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)} - \frac{s^{2-2\alpha}(\phi'(s))^2 r(s)}{4\phi(s)\delta_1(s,t_2)}]s^{\alpha-1}ds > \int_{t_3}^{t}(\frac{1}{s} - \frac{1}{4s})ds = \int_{t_3}^{t}\frac{3}{4s}ds \to \infty\ (t\to\infty).$$

From the analysis above one can see [14, (2.6)-(2.9)] all hold. So it follows from [14, Corollary 2.2] that every solution of Eq. (2.1) is oscillatory or tends to zero.

2.2 Fractional difference equation

Next we consider the following fractional difference equation:

$$\Delta^{(\frac{1}{2})}\{\sqrt{t}\Delta^{(\frac{1}{2})}[t^{-\frac{1}{2}}\Delta^{(\frac{1}{2})}x(t)]\} + t^{-\frac{5}{2}}\Delta^{(\frac{1}{2})}[t^{-\frac{1}{2}}\Delta^{(\frac{1}{2})}x(t)] + t^{-\frac{3}{2}}x(t) = 0,\ t \in [2,\infty)_{\mathbb{Z}}, \tag{2.2}$$

where $\Delta^{(\frac{1}{2})}$ denotes the fractional difference operator of order $\frac{1}{2}$.

Related to (1.1), one has $\mathbb{T} = \mathbb{Z},\ \alpha = \frac{1}{2},\ a(t) = \sqrt{t},\ p(t) = t^{-\frac{5}{2}},\ q(t) = t^{-\frac{3}{2}},\ \widetilde{p}(t) = t^{-\frac{1}{2}}p(t) = t^{-3},\ r(t) = t^{-\frac{1}{2}},\ t_0 = 2$. Then $\mu(t) = \sigma(t) - t = 1$, and

$$1-\mu(t)\frac{\widetilde{p}(t)}{a(t)}=1-t^{-\frac{7}{2}}\geq 1-t^{-3}\geq 1-\frac{1}{2^3}>0,$$

which means $-\frac{\widetilde{p}}{a}\in\Re_+$. So according to [16, Lemma 2] one can obtain that

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0)=e_{-\frac{\widetilde{p}}{a}}(t,2)\geq 1-\int_2^t\frac{\widetilde{p}(s)}{a(s)}\Delta s=1-\int_2^t s^{-\frac{7}{2}}\Delta s=1-\sum_{s=2}^{t-1}s^{-\frac{7}{2}}$$
$$\geq 1-\int_1^{t-1}s^{-\frac{7}{2}}ds=1+\frac{2}{5}[(t-1)^{-\frac{5}{2}}-1]>\frac{3}{5},$$

and

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0)\leq\exp(-\int_2^t\frac{\widetilde{p}(s)}{a(s)}\Delta s)<1.$$

To use [14, Corollary 2.3], one needs to verify [14, (2.10)-(2.13)]. To this end, one has

$$\sum_{s=t_0}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}=\sum_{s=2}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,2)}{a(s)}s^{\alpha-1}=\sum_{s=2}^{\infty}\frac{e_{-\frac{\widetilde{p}}{a}}(s,2)}{s}>\frac{3}{5}\sum_{s=2}^{\infty}\frac{1}{s}=\infty,$$

and

$$\sum_{s=t_0}^{\infty}\frac{1}{r(s)}s^{\alpha-1}=\sum_{s=2}^{\infty}1=\infty.$$

Furthermore,

$$\sum_{\xi=t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)})]=\sum_{\xi=t_0}^{\infty}[\frac{\xi^{\alpha-1}}{r(\xi)}\sum_{\tau=\xi}^{\infty}\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,2)}{a(\tau)}\sum_{s=\tau}^{\infty}\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(s+1,2)})]$$
$$>\frac{3}{5}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}(\frac{1}{\tau}\sum_{s=\tau}^{\infty}\frac{1}{s^2})]>\frac{3}{5}\sum_{\xi=2}^{\infty}[\sum_{\tau=\xi}^{\infty}(\frac{1}{\tau}\sum_{s=\tau}^{\infty}\frac{1}{s(s+1)})]=\frac{3}{5}\sum_{\xi=2}^{\infty}\sum_{\tau=\xi}^{\infty}\frac{1}{\tau^2}$$
$$>\frac{3}{5}\sum_{\xi=2}^{\infty}\sum_{\tau=\xi}^{\infty}\frac{1}{\tau(\tau+1)}=\frac{3}{5}\sum_{\xi=2}^{\infty}\frac{1}{\xi}=\infty.$$

So [14, (2.10)-(2.12)] hold. Moreover, since for a sufficiently large t_2, it holds that

$$\delta_1(t,t_2)=\sum_{s=t_2}^{t-1}\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}>\frac{3}{5}\sum_{s=t_2}^{t-1}\frac{1}{s}\to\infty\ (t\to\infty),$$

then there exists $t_3>t_2$ such that $\delta_1(t,t_2)>1$ for $t\in[t_3,\infty)_{\mathbb{Z}}$. If we let $\phi(t)=t$, $\varphi(t)=0$ in [14, (2.13)], then one can obtain that

$$\sum_{s=t_3}^{t-1}[q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(s+1,t_0)}-\frac{s^{2-2\alpha}(\phi(s+1)-\phi(s))^2r(s)}{4\phi(s)\delta_1(s,t_2)}]s^{\alpha-1}>\sum_{s=t_3}^{t-1}(\frac{1}{s}-\frac{1}{4s})=\sum_{s=t_3}^{t-1}\frac{3}{4s}\to\infty\ (t\to\infty).$$

So [14, (2.13)] also holds. After an application of [14, Corollary 2.3] one can see that every solution of Eq. (2.2) is oscillatory or tends to zero.

2.3 Fractional $q-$ difference equation

Finally we consider the following fractional $q-$ difference equation:

$$\Delta^{(\frac{3}{5})}\{t^{0.6}\Delta^{(\frac{3}{5})}[t^{-0.4}\Delta^{(\frac{3}{5})}x(t)]\}+t^{-2.4}\Delta^{(\frac{3}{5})}[t^{-0.4}\Delta^{(\frac{3}{5})}x(t)]+t^{-1.6}x(t)=0,\ t\in[\beta,\infty)_{\beta^{\mathbb{Z}}},\tag{2.3}$$

where $\Delta^{(\frac{3}{5})}$ denotes the fractional difference operator of order $\frac{3}{5}$, $\beta\geq 2$.

Related to (1.1), one has $\mathbb{T}=\beta^{\mathbb{Z}}$, $\alpha=\frac{3}{5}$, $a(t)=t^{0.6}$, $p(t)=t^{-2.4}$, $q(t)=t^{-1.6}$, $\widetilde{p}(t)=t^{-0.4}p(t)=$

$t^{-2.8},\ r(t)=t^{-0.4},\ t_0=\beta$. Then $\mu(t)=\sigma(t)-t=t(\beta-1)$, and considering $t\geq\beta$, one has

$$1-\mu(t)\frac{\widetilde{p}(t)}{a(t)}=1-t(\beta-1)\frac{1}{t^{3.4}}=1-(\beta-1)\frac{1}{t^{2.4}}\geq 1-(\beta-1)\frac{1}{\beta^2}=\frac{\beta^2-\beta+1}{\beta^2}>0,$$

which means $-\frac{\widetilde{p}}{a}\in\mathfrak{R}_+$. So we obtain

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0)=e_{-\frac{\widetilde{p}}{a}}(t,\beta)\geq 1-\int_\beta^t\frac{\widetilde{p}(s)}{a(s)}\Delta s=1-\int_\beta^t\frac{1}{s^{3.4}}\Delta s\geq 1-\int_\beta^t\frac{1}{s^3}\Delta s=1-(\beta-1)\frac{t^{-2}-\beta^{-2}}{\beta^{-2}-1}$$
$$=\frac{1+(\beta-1)t^{-2}-\beta^{-1}}{1-\beta^{-2}}>\frac{1-\beta^{-1}}{1-\beta^{-2}}\geq\frac{1}{2-2\beta^{-2}}=\frac{\beta^2}{2(\beta^2-1)},$$

and

$$e_{-\frac{\widetilde{p}}{a}}(t,t_0)\leq\exp(-\int_q^t\frac{\widetilde{p}(s)}{a(s)}\Delta s)<1.$$

Now we verify the following conditions:

$$\int_{t_0}^\infty\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s=\int_\beta^\infty\frac{e_{-\frac{\widetilde{p}}{a}}(s,\beta)}{a(s)}s^{\alpha-1}\Delta s=\int_\beta^\infty\frac{e_{-\frac{\widetilde{p}}{a}}(s,\beta)}{s}\Delta s>\frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty\frac{1}{s}\Delta s=\infty,$$

and

$$\int_{t_0}^\infty\frac{1}{r(s)}\Delta^\alpha s=\int_{t_0}^\infty\frac{1}{r(s)}s^{\alpha-1}\Delta s=\int_{t_0}^\infty 1\Delta s=\infty.$$

Furthermore,

$$\int_{t_0}^t[\frac{1}{r(\xi)}\int_\xi^\infty(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty\frac{q(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta^\alpha s)\Delta^\alpha\tau]\Delta^\alpha\xi$$
$$=\int_{t_0}^\infty\xi^{\alpha-1}[\frac{1}{r(\xi)}\int_\xi^\infty\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,t_0)}{a(\tau)}\int_\tau^\infty\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}\Delta s)\Delta\tau]\Delta\xi$$
$$=\int_\beta^\infty\xi^{\alpha-1}[\frac{1}{r(\xi)}\int_\xi^\infty\tau^{\alpha-1}(\frac{e_{-\frac{\widetilde{p}}{a}}(\tau,\beta)}{a(\tau)}\int_\tau^\infty\frac{q(s)s^{\alpha-1}}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),\beta)}\Delta s)\Delta\tau]\Delta\xi$$
$$>\frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty(\frac{1}{\tau}\int_\tau^\infty\frac{1}{s^2}\Delta s)\Delta\tau]\Delta\xi>\frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty(\frac{1}{\tau}\int_\tau^\infty\frac{1}{s\sigma(s)}\Delta s)\Delta\tau]\Delta\xi$$
$$=\frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty(\frac{1}{\tau}[-\frac{1}{s}]_\tau^\infty)\Delta\tau]\Delta\xi=\frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty\frac{1}{\tau^2}\Delta\tau]\Delta\xi>\frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty[\int_\xi^\infty\frac{1}{\tau\sigma(\tau)}\Delta\tau]\Delta\xi$$
$$=\frac{\beta^2}{2(\beta^2-1)}\int_\beta^\infty\frac{1}{\xi}\Delta\xi=\infty.$$

On the other hand, one can see for a sufficiently large t_2 that

$$\delta_1(t,t_2)=\int_{t_2}^t\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}\Delta^\alpha s=\int_{t_2}^t\frac{e_{-\frac{\widetilde{p}}{a}}(s,t_0)}{a(s)}s^{\alpha-1}\Delta s>\frac{\beta^2}{2(\beta^2-1)}\int_{t_2}^t\frac{1}{s}\Delta s\to\infty\ (t\to\infty).$$

So there exists $t_3>t_2$ such that $\delta_1(t,t_2)>1$ for $t\in[t_3,\infty)_{q^{\mathbb{Z}}}$.

To use [15, Theorem 2.2], let $m=1,\ \phi(t)=t,\ \varphi(t)=0$ in [15, (2.4)], and one has

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)}\{\int_{t_3}^t[(t-s)q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\frac{(\phi^{(\alpha)}(s))^2r(s)}{4\phi(s)\delta_1(s,t_2)}]\Delta^\alpha s\}$$
$$=\limsup_{t\to\infty}\frac{1}{(t-t_0)}\{\int_{t_3}^t[(t-s)q(s)\frac{\phi(s)}{e_{-\frac{\widetilde{p}}{a}}(\sigma(s),t_0)}-\frac{s^{2-2\alpha}(\phi^\Delta(s))^2r(s)}{4\phi(s)\delta_1(s,t_2)}]s^{\alpha-1}\Delta s\}$$

$$> \lim_{t\to\infty} \sup \frac{1}{(t-t_0)} \int_{t_3}^{t} (t-s)\frac{3}{4s}\Delta s = \lim_{t\to\infty} \sup[\frac{t}{(t-\beta)} \int_{t_3}^{t} \frac{3}{4s}\Delta s - \frac{3(t-t_2)}{4(t-\beta)}] = \infty,$$

which means [15, (2.4)] also holds, and by [15, Theorem 2.2] one can deduce that every solution of Eq. (2.3) is oscillatory or tends to zero.

3 Conclusions

In this paper, we have presented some applications for the oscillation criteria for certain fractional differential equations, fractional difference equations and fractional $q-$ difference equations. The validity of the established results are illustrated by three corresponding examples.

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