On multipliers of hyper MV-algebras

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Abstract

In the paper, we introduce the notion of multipliers of a hyper MV-algebra in the paper and investigate some properties of hyper MV-algebras in terms of multipliers. In addition, a set of equivalent conditions are established for two multipliers of a hyper MV-algebra to be equal in the sense of mappings. Further, some propositions of multipliers related with $Fix_d(M)$ are given.

Keywords: hyper MV-algebra; multiplier; hyper isotone; hyper regular multiplier

1. Introduction

Algebraic hyperstructures were introduced in 1934 by Marty [1] at 8th Congress of Scandinavian Mathematicians, which represent a natural extension of classical algebraic structures.Ghorbani et al. [2] applied the hyper structure to MV-algebras and introduced the concept of hyper MV-algebra which is a generalization of MV-algebra and investigated some related results. Torkzadeh and Ahadpanah [3] defined some hyper operations on hyper MV-algebras and introduced the notion of (weak)hyper MV-ideals in hyper MV-algebras. In order to find conditions that a hyper MV-algebra becomes an MV-algebra, Torkzadeh and Ghorbani [4] characterized hyper MV -algebras in which 0 is a scalar element. Xin et al. [5] introduced the notions of Riečan states and Bosbach states on a hyper MV-algebra and derive some properties of them. Hyperstructures were also applied other algebraic structures in recent years, such as hyper effect algebras [6, 7], hyper hoop-algebras [8] and hyper BCK-algebras [9], and some important results were obtained, such as

The concept of multiplier for distributive lattices was defined by Cornish [10]. Multipliers are used in order to give a non standard construction of the maximal lattice of quotients for a distributive lattice [11]. Tayebi Khorami and Borumand Saeid [12] introduced the notion of multiplier in BL-algebra and study relationships between multipliers and some special mappings, likeness closure operators, homomorphisms and $(\otimes, \vee)$-derivations in BL-algebras. [13] gave a necessary and sufficient condition for a Lau type binary operation defined by two mappings to be an algebra-operation in terms of multipliers. In 2013, Rao [14] introduced the notion of multipliers in a hypersemilattice and studied some properties of multipliers. Uzay and Firat [15] introduced the notion of multipliers of a hyper BCI-algebra, and discuss some properties of hyper BCI-algebras.

Inspired by [14, 15], we introduce the notion of multipliers of a hyper MV-algebra in the paper. Then we investigate some properties of hyper MV-algebras in terms of multipliers. In addition, a set of equivalent conditions are established for two multipliers of a hyper MV-algebra to be equal in the sense of mappings. Also we gave some propositions of multipliers related with $Fix_d(M)$.

2. Preliminaries

To facilitate our discussion, we give some primary notions and previous results about MV-algebras and hyper MV-algebras in this section, which are necessary for the subsequent discussions.

An algebra $(M, \oplus, \neg, 0)$ of type $(2, 1, 0)$ is called an MV-algebra if it satisfies the following axioms: for any $x, y, z \in M$,

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(MV2) $x \oplus y = y \oplus x$,

(MV3) $x \oplus 0 = x$,
(MV4) $(x^*)^* = x$,
(MV5) $x \oplus 0^* = 0^*$,
(MV6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

A partial order $\leq$ introduced on an MV-algebra M by defining $x \leq y$ if $x^* \oplus y = 0^*$, which makes A into a distributive lattice.

A hyperoperation on a nonempty set A is a mapping $\circ : A \times A \to P^\star(A) \backslash \{\emptyset\}$, where $P^\star(A)$ is the set of all the nonempty subsets of A. The couple $(A, \circ)$ is called a hypergroupoid.

Definition 2.1. [2] A hyper MV-algebra $(M, \oplus, *, 0)$ is a non-empty set M endowed with a binary hyper operation $\oplus$, a unary operation $*$ and a constant 0 satisfying the following conditions: for any $x, y, z \in M$,

(hMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(hMV2) $x \oplus y = y \oplus x$,
(hMV3) $(x^*)^* = x$,
(hMV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(hMV5) $0^* \in x \oplus 0^*$,
(hMV6) $0^* \in x \oplus x^*$,
(hMV7) if $x \ll y$ and $y \ll x$, then $x = y$,

where $x \ll y$ is defined by $0^* \in x^* \oplus y$ and for every $A, B \subseteq M$, if there exist $a \in A$ and $b \in B$ such that $a \ll b$, then we define $A \ll B$. We define $1 := 0^*$, $x \otimes y := (x^* \oplus y^*)^*$, $x \ominus y := x \otimes y^* = (x^* \oplus y)^*$, $x \to y := (x \otimes y^*)^*$, and for every $A \subseteq M$, $A^* = \{a^* | a \in A\}$. Obviously, $x \ll y$ iff $0 \in x \ominus y$ or $1 \in x \to y$.

In what follows, unless otherwise specified, we denote a hyper MV-algebra $(M, \oplus, *, 0)$ by M. Let M be a a hyper MV-algebra. If A and B are nonempty subsets of M, we denote: for any $a, b, x \in M$,

(1) $x \circ A = \{x\} \circ A = \bigcup\limits_{a \in A} x \circ a$, $A \circ x = A \circ \{x\} = \bigcup\limits_{a \in A} a \circ x$,
(2) $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$,

where $\circ \in \{\oplus, \ominus, \to\}$.

There exists a connection between hyper MV-algebras and Mv-algebras. MV-algebras are special cases of hyper MV-algebras. In fact, we have

(1) If M is a hyper MV-algebra such that $x \oplus y$ is a singleton for any $x, y \in M$, then M is an MV-algebra.
(2) Let M be an MV-algebra. Define a hyper operation $\oplus'$ on M by $x \oplus' y = \{t \in M | 0 \leq t \leq x \oplus y\} = [0, x \oplus y]$ for any $x, y \in M$. Then $(M, \oplus', *, 0)$ is a hyper MV-algebra.

Now we state some primary properties of hyper MV-algebras in the following proposition.

Proposition 2.2. [2**?** , 3] Given a hyper MV-algebra $(M, \oplus, *, 0)$, for any $x, y, z \in M$ and for any nonempty subsets A, B and C of M, the following results hold:

(1) $0 \ll x \ll 1$, $x \ll x$,
(2) $x \ll x \oplus y$, $A \ll A \oplus B$,
(3) $x \in x \oplus 0$, $x \in x \otimes 1$, $0 \in x \otimes x^*$, $0 \in x \otimes 0$;
(4) $x \in 1 \to x$, $1 \in x \to x$, $1 \in (1 \to x) \to x$;
(5) $0 \oplus 0 = \{0\}$, $1 \otimes 1 = \{1\}$, $1 \to 0 = \{0\}$;
(6) $x \otimes y \ll x \ll x \oplus y$ and $A \otimes B \ll A \ll A \oplus B$;
(7) $(x \to y) \ll y$ and $(A \to B) \ll B$;
(8) $(A^*)^* = A$ and $A \to (B \to C) = B \to (A \to C)$;
(9) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$, $(A \otimes B) \otimes C = A \otimes (B \otimes C)$;
(10) $x \ll y$ implies $y^* \ll x^*$, and $A \ll B$ implies $B^* \ll A^*$;
(11) $x \ll y$ implies $x \oplus z \ll y \oplus z$ and $x \otimes z \ll y \otimes z$;

(12) $A \subseteq B$ implies $A \ll B$;
(13) $A \ll B$ and $B \ll C$ implies $A \ll C$;
(14) if $y \in x \oplus 0$, then $y \ll x$;
(15) $x \otimes y \ll z$ if and only if $x \ll y \to z$.

Definition 2.3. [2] Let M_1 and M_2 be two hyper MV-algebras. A mapping $f : M_1 \to M_2$ is said to be a homomorphism, for any $x, y \in M_1$, if

$$f(0) = 0,\ f(x \oplus y) = f(x) \oplus (y) \text{ and } f(x^*) = f(x)^*$$

where for any $A \subseteq M_1$, $f(A) := \{f(a)|a \in A\}$.

3. Multipliers of hyper MV-algebras

In this section, we define the notion of multiplier on a hyper MV -algebra, and present some characterizations of a multiplier.

Definition 3.1. Let M be a hyper MV-algebra, and $d : M \to M$ be a function. Then d is said to be a multiplier of M if it satisfies: for any $x, y \in M$,

$$d(x \otimes y) = d(x) \otimes y.$$

Example 3.2. Let $M = \{0, b, 1\}$. Define a hyperoperation "$\oplus$" on M by the following table:

$\oplus$	0	b	1
0	{0}	{0,b}	{1}
b	{0,b}	{0,b,1}	{0,b,1}
1	{1}	{0,b,1}	{1}

Define a unary operation $*$ as follows: $0^* = 1,\ 1^* = 0, b^* = b$. Then $(M, \oplus, *, 0)$ is a hyper MV-algebra. And we have the following multiplication table:

$\otimes$	0	b	1
0	{0}	{0,b,1}	{0}
b	{0,b,1}	{0,b,1}	{b,1}
1	{0}	{b,1}	{1}

Define a map $d : M \to M$ as follows:

$$d(x) = \begin{cases} 0, & x = 0, \\ a, & x = b, \\ b, & x = 1. \end{cases}$$

Route calculations show that d is a multiplier of M.

Proposition 3.3. Let d be a multiplier of a hyper MV-algebra M. Then for any $x, y \in M$,

(1) $d(x \ominus y) = d(x) \ominus y$;
(2) $d(x \ominus d(x)) \ll 0$;
(3) $x \otimes d(y) = d(x) \otimes y$.

Proof. (1) For any $x, y \in M$, $d(x \ominus y) = d(x \otimes y^*) = d(x) \otimes y^* = d(x) \ominus y$.

(2) According to (1), we have $d(x \ominus d(x)) = d(x) \ominus d(x)$. Since $0 \in d(x) \ominus d(x) = d(x) \otimes (d(x))^*$ and $0 \ll 0$, then $d(x \ominus d(x)) \ll 0$.

(3) $x \otimes d(y) = d(y) \otimes x = d(y \otimes x) = d(x \otimes y) = d(x) \otimes y$. □

Proposition 3.4. Let d be a multiplier of a hyper MV-algebra M. If d is idempotent, that is, $d^2(x) = d(x)$ for any $x \in M$, then $d(x \otimes y) = d(x) \otimes d(y)$, for any $x, y \in M$.

Proof. Suppose that d is an idempotent multiplier of M. For any $x, y \in M$,

$$d(x \otimes y) = d^2(x \otimes y) = d(d(x \otimes y)) = d(d(x) \otimes y) = d(y \otimes d(x)) = d(y) \otimes d(x) = d(x) \otimes d(y).$$

□

Definition 3.5. Let M be a hyper MV-algebra and $d : M \to M$ be a self-mapping.

(1) d is said to be regular if $d(0) = 0$;
(2) d is hyper isotone if $x \ll y$ implies $d(x) \ll d(y)$ for any $x, y \in M$.

Proposition 3.6. Let d be a regular multiplier of a hyper MV-algebra M. Then for any $x, y \in M$,

(1) $d(x) \ll x$;
(2) $d(x) \otimes d(y) \ll d(x \otimes y) \ll x \otimes y$;
(3) $d(x \ominus y) \ll d(x) \ominus d(y)$.
(4) $d(x \ominus y) \ll x \ominus y$.

Proof. (1) Since $0 = d(0) \in d(x \otimes x^*) = d(x) \otimes x^*$, thus $d(x) \ll x$.
(2) According to (1) and Proposition 2.2, we have $d(x) \otimes d(y) \ll x \otimes d(y) = d(x \otimes y) \ll x \otimes y$.
(3) Since $d(y) \ll y$, then $y^* \ll (d(y))^*$ by Proposition 2.2, and so

$$d(x \ominus y) = d(x) \ominus y = d(x) \otimes y^* \ll d(x) \otimes (d(y))^* = d(x) \ominus d(y)$$

(4) Since $d(x) \ll x$, then $d(x \ominus y) = d(x) \ominus y \ll x \ominus y$. □

Proposition 3.7. Let d be a regular multiplier of a hyper MV-algebra M. If d is an endomorphism, then d is hyper isotone.

Proof. Let $x, y \in M$ such that $x \ll y$. It follows that $0 \in x \ominus y$. Since d is both a regular multiplier and an endomorphism, then $0 = d(0) \in d(x \ominus y) = d(x) \ominus d(y)$, and so, $d(x) \ll d(y)$. Thus, d is hyper isotone. □

Proposition 3.8. Let d_1 and d_2 be two multipliers of a hyper MV-algebra M. Then $d_1 \cdot d_2$ is also a multiplier of M, where $(d_1 \cdot d_2)(x) = d_1(d_2(x))$ for any $x \in M$.

Proof. Since d_1 and d_2 are multipliers of M, it follows that

$$(d_1 \cdot d_2)(x \otimes y) = d_1(d_2(x) \otimes y) = d_1(d_2(x)) \otimes y = (d_1 \cdot d_2)(x) \otimes y$$

for any $x, y \in M$. Thus, $d_1 \cdot d_2$ is a multiplier of M. □

Definition 3.9. Let d be a multiplier of a hyper MV-algebra M. Define

$$Fix_d(M) = \Delta_d(M) \cup (\textstyle\bigcup_{i \in I} A_i),$$

where $\Delta_d(M) := \{x \in M | d(x) = x\}$ and $d(A_i) = A_i$ for any $A_i \subseteq M$.

If d is an idempotent multiplier, then clearly $d(x) \in \Delta_d(M)$ for any $x \in M$.

Proposition 3.10. Let d be a multiplier of a hyper MV-algebra M. Then we have

(1) $\Delta_d(M) \subseteq \Delta_{d^2}(M)$;
(2) If $x \in M$ and $y \in \Delta_d(M)$, then $d(x \ominus y) = d(x) \ominus d(y)$;
(3) If $x \in \Delta_d(M)$ and $y \in M$, then $x \otimes y \subseteq Fix_d(M)$ and $x \ominus y \subseteq Fix_d(M)$.

Proof. (1) For any $x \in \Delta_d(M)$, we have $x = d(x)$. It follows that $d^2(x) = d(d(x)) = d(x) = x$, and so $x \in \Delta_{d^2}(M)$, therefore $\Delta_d(M) \subseteq \Delta_{d^2}(M)$.

(2) If $x \in M$ and $y \in \Delta_d(M)$, then $d(x \ominus y) = d(x) \ominus y = d(x) \ominus d(y)$.

(3) If $x \in \Delta_d(M)$ and $y \in M$, we have $d(x \otimes y) = d(x) \otimes y = x \otimes y$, and so $x \otimes y \subseteq Fix_d(M)$. It follows from Proposition 3.3 that $d(x \ominus y) = d(x) \ominus y = x \ominus y$, and therefore $x \ominus y \subseteq Fix_d(M)$. □

Theorem 3.11. Let d_1 and d_2 be two idempotent multipliers of a hyper MV-algebra M such that $d_1 \cdot d_2 = d_2 \cdot d_1$. Then the following conditions are equivalent:

(1) $d_1 = d_2$;
(2) $d_1(M) = d_2(M)$;
(3) $\Delta_{d_1}(M) = \Delta_{d_2}(M)$.

Proof. (1) ⇒ (2) Obviously.

(2) ⇒ (3) Assume that $d_1(M) = d_2(M)$. For any $x \in \Delta_{d_1}(M)$, we have $x = d_1(x) \in d_1(M) = d_2(M)$. Then there exists $y \in M$ such that $x = d_2(y)$. And $d_2(x) = d_2(d_2(y)) = d_2(y) = x$, so $x \in \Delta_{d_2}(M)$, it follows that $\Delta_{d_1}(M) \subseteq \Delta_{d_2}(M)$. Similarly, we can prove $\Delta_{d_2}(M) \subseteq \Delta_{d_1}(M)$. Thus $\Delta_{d_1}(M) = \Delta_{d_2}(M)$.

(3) ⇒ (1) Suppose that $\Delta_{d_1}(M) = \Delta_{d_2}(M)$. For any $x \in M$, since $d_1(x) \in \Delta_{d_1}(M) = \Delta_{d_2}(M)$, we get that $d_2(d_1(x)) = d_1(x)$. Similarly, $d_1(d_2(x)) = d_2(x)$. Then $d_1(x) = d_2(d_1(x)) = (d_2 \cdot d_1)(x) = (d_1 \cdot d_2)(x) = d_1(d_2(x)) = d_2(x)$, thus, $d_1 = d_2$. □

Let $(M_1, \oplus_1, *_1, 0_1)$, $(M_2, \oplus_2, *_2, 0_2)$ be two hyper MV-algebras, and $M = M_1 \times M_2$. We define a hyper operation $\oplus$, a unary operation $*$ and a constant 0 on M as follows:

$$\begin{aligned}(a_1, a_2) \oplus (b_1, b_2) &= (a_1 \oplus_1 b_1, a_1 \oplus_2 b_2),\\ (a_1, a_2)^* &= (a_1^{*_1}, a_2^{*_2}),\\ 0 &= (0_1, 0_2),\end{aligned}$$

for any $(a_1, a_2), (b_1, b_2) \in M_1 \times M_2$, and a hyperorder $\ll$ on M by

$$(a_1, a_2) \ll (b_1, b_2) \text{ iff } a_1 \ll_1 b_1, a_2 \ll_2 b_2.$$

Then $(M, \oplus, *, 0)$ is a hyper MV-algebra, and we call it the product of hyper MV-algebras M_1 and M_2.

Proposition 3.12. Let $(M, \oplus', *', 0')$ be a hyper MV-algebra, d_1 and d_2 be multipliers of M_1. Then the product $d_1 \times d_2$ of d_1 and d_2 is a multiplier of $M \times M$, where

$$(d_1 \times d_2)(x, y) := d_1(x) \otimes' d_2(y),$$

for any $(x, y) \in M \times M$.

Proof. For any $x_1, x_2, y_1, y_2 \in M$, we get that

$$\begin{aligned}(d_1 \times d_2)((x_1, y_1) \otimes (x_2, y_2)) &= (d_1 \times d_2)((x_1 \otimes' x_2, y_1 \otimes' y_2)\\ &= d_1(x_1 \otimes' x_2) \otimes' d_2(y_1 \otimes' y_2)\\ &= d_1(x_1) \otimes' x_2 \otimes' d_2(y_1) \otimes' y_2\\ &= (d_1(x_1) \otimes' d_2(y_1)) \otimes' (x_2 \otimes' y_2)\\ &= (d_1 \times d_2)(x_1, y_1)) \otimes (x_2, y_2).\end{aligned}$$

Hence $d_1 \times d_2$ is a multiplier of $M \times M$. □

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