A Note on Solving Nonlinear Differential Equations Using Modified Laplace Decomposition Method

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Abstract

In the field of Applied Mathematics, physics, and engineering, to explain phenomena occurring in these fields, models are developed in the form of differential equations. Many of these phenomena are typically represented as nonlinear differential equations. While there exist a handful of analytical methods to solve some regular problems, often an analytical solution turns out to be quite difficult to attain using traditional methods. Therefore the objective is to explore a numerical method or a semi-analytical method that yields the best approximation. In this paper, we investigate a consistent modification of Laplace decomposition method using Adomian polynomials to solve nonlinear ordinary and partial differential equations. The method is introduced and to further demonstrate its effectiveness, it is applied to solve three differential equations where nonlinearity appears in different forms.

Keywords: Nonlinear PDE, Modified Laplace Decomposition Method.

1 Introduction

The Adomian decomposition method (ADM) is extremely useful in solving complex nonlinear partial differential equations (NLPDE). As we observe in the fields of physics, acoustics, plasma physics, and fluid dynamics, many problems in these fields can be modeled by NLPDEs. In the nonlinear case for ordinary differential equations (ODE) and PDE, ADM can successfully solve equations without breaking them into little ones. The method avoids linearization, discretization and other unrealistic assumptions. In solving a general nonlinear PDE or ODE, we first take the Laplace transform of the entire equation. The nonlinear term is then replaced by a series of Adomian Polynomials. The evaluation of these polynomials is needed, as they contribute to the solutions series components. The crucial aspect of the method is employment of the "Adomian polynomials" which allows for solution convergence of the nonlinear part of the equation, without simply linearizing the system. These polynomials mathematically generalize to a Maclaurin series about an arbitrary external parameter; which gives this method more flexibility than direct Taylor series expansion. The modified Laplace decomposition method is much easier to implement as compared to the Adomian decomposition method where huge complexities are involved [4]. This method has also been demonstrated to be effective for the study of boundary layer equations [5].

This technique basically illustrates how the Laplace transform can be used to approximate the solutions of the nonlinear differential equations by manipulating the decomposition method which was first introduced by Adomian [3]. The method is well suited to physical problems since it does not require unnecessary linearization, perturbation and other restrictive methods and assumptions which may change the problem being solved to a great extent. Laplace Adomian decomposition method (LADM) was first proposed by Suheil A. Khuri, and has been successfully employed to find the solution of differential equations. The major advantage of this method is its capability of combining the two powerful methods to obtain exact solutions for nonlinear equations. However, LADM will generate "noise term" for inhomogeneous equations [6]. Therefore, M. Hussain developed a modified Laplace decomposition method (MLDM) which can accelerate the rapid convergence of series solution when compared with Laplace Adomian decomposition method [2]. Earlier Adomian and Rach introduced the phenomena of the so-called noise terms in [3]. The "noise terms" are defined as the identical terms with opposite signs that appear in the components of the series solution of $u(x)$. In [1], it is concluded that if terms in the component u_0 are canceled by terms in the component u_1, even though u_1 contains further terms, then the remaining non-canceled terms of u_1 provide the exact solution. It is suggested that the noise terms appear always for inhomogeneous equations. The necessary condition for the"noise terms" to appear in the components u_0 and u_1 is that the exact solution must appear as the part of u_0 among other terms. This is true for only a special kind of inhomogeneous equation. It is also evident when you apply the MLDM, to be mindful where one separates the u terms after applying the Laplace transform for example rather you choose to leave all the single u terms on one side or separate them on both sides, for this may affect the result of the numerical solution; this will be demonstrated in example 1. Many authors have modified the Laplace Decomposition method to solve different nonlinear equations in order to speed up the convergence of the series solution [6-9]. In this paper, we will approximate a numerical solution and ODE, and PDE using u_n and then compare our numerical solution to a known exact solution, for some cases, the method can give us an exact solution. With the fast convergence capabilities of Adomian polynomials, we should expect to use only a few polynomials to effectively approximate our numerical solution. In this work, we will use the modified form of Laplace decomposition method introduced by Khuri, and adopted by Hussain and Khan [1]. This numerical technique basically illustrates how the Laplace transform can be employed to approximate the solutions of the NLPDE by manipulating the decomposition method. Here, modified Laplace decomposition is implemented to nonlinear ordinary and partial differential equations as well as a nonlinear system of PDEs. The effectiveness and the usefulness of modified Laplace decomposition method are demonstrated by comparing the graphs of the numerical solutions to the exact solutions of these two models wherever applicable. For the cases which yield the exact solution, graphical representation of comparisons between exact and approximating solutions are omitted.

2 Description of the Method

The objective of this section is to discuss the use of modified Laplace transform method for the nonlinear partial differential equations. For suitability, we deliberate the general form of second order nonhomogeneous nonlinear partial differential equations with initial conditions and boundary conditions as given below:

$$Qu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t), \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x),$$

where Q is a second order differential operator, R is the remaining linear operator, N is a general nonlinear operator, and, $h(x,t)$ is a source term. We are only applying this method to any ODE or PDE to the order of two. We begin exercising this method by applying the Laplace transform on both sides of the given equation as shown below:

$$\mathcal{L}[Qu(x,t)] + \mathcal{L}[Ru(x,t)] + \mathcal{L}[Nu(x,t)] = \mathcal{L}[h(x,t)].$$

Utilizing the differentiation property of the Laplace transform we get,

$$s^2\mathcal{L}[u(x,t)] - sf(x) - g(x) + \mathcal{L}[Ru(x,t)] + \mathcal{L}[Nu(x,t)] = \mathcal{L}[h(x,t)],$$
$$\mathcal{L}[u(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}\mathcal{L}[h(x,t)] - \frac{1}{s^2}\Big(\mathcal{L}[Ru(x,t)] + \mathcal{L}[Nu(x,t)]\Big)$$

Using the superposition principle, the solution can be represented as an infinite series e.g., $\sum_{n=0}^{\infty} u_n(x,t)$. If we observe the nonlinear operator $Nu(x,t)$ from our equation, we decompose it as a series of Adomian polynomials as $Nu(x,t) = \sum_{n=0}^{\infty} A_n$, where A_n are Adomian polynomials of u_n and it can be determined by the relation

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[N\sum_{i=0}^{\infty}\lambda^n u_i\Big]_{\lambda=0}, \qquad n = 0, 1, 2, \ldots$$

Using our previous equations, we can recast

$$\mathcal{L}\Big[\sum_{n=0}^{\infty}u_n(x,t)\Big] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}\mathcal{L}[h(x,t)] - \frac{1}{s^2}\Big(\mathcal{L}[Ru(x,t)]\Big) - \frac{1}{s^2}\mathcal{L}\Big[\sum_{n=0}^{\infty}A_n\Big].$$

When we compare both sides of the equations in terms of u_0, u_1, u_{n+1}, we have

$$\mathcal{L}[u_0(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}\mathcal{L}[h(x,t)],$$
$$\mathcal{L}[u_1(x,t)] = -\frac{1}{s^2}\mathcal{L}[Ru_0(x,t)] - \frac{1}{s^2}\mathcal{L}[A_0],$$
$$\mathcal{L}[u_2(x,t)] = -\frac{1}{s^2}\mathcal{L}[Ru_1(x,t)] - \frac{1}{s^2}\mathcal{L}[A_1].$$

The preceding terms, depending on how far of the approximation can be determined by the given recursive relation,

$$\mathcal{L}[u_{n+1}(x,t)] = -\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)] - \frac{1}{s^2}\mathcal{L}[A_n], \qquad n \geq 1.$$

If we apply the inverse Laplace transform to the above equation, we get

$$u_{n+1}(x,t) = -\mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)] + \frac{1}{s^2}\mathcal{L}[A_n]\Big], \qquad n \geq 0.$$

We assume, $u_n(x,t) = K(x,t) = K_0(x,t) + K_1(x,t),\ n < 2$. Under this assumption, we propose a slight variation only in the components u_0, u_1. The variation we suggest is that only the part $K_0(x,t)$ be assigned to the u_0, whereas the remaining part $K_1(x,t)$ be combined with the other terms. Incorporating these proposals, we get a powerful modified recursive algorithm as follows below:

$$u_1(x,t) = K_1(x,t) - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[Ru_0(x,t)] + \frac{1}{s^2}\mathcal{L}[A_0]\Big],$$
$$u_{n+1}(x,t) = -\mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)] + \frac{1}{s^2}\mathcal{L}[A_n]\Big], \quad n \geq 1.$$

The solution through the modified Adomian decomposition method is highly depend upon the choice of $K_0(x,t)$ and $K_1(x,t)$. The initial solution is important, as the initial solution always leads to noise oscillation during the iteration procedure.

3 Numerical Validation and Discussions

To demonstrate the effectiveness of this method to approximating nonlinear ordinary and partial differential equations, we shall take two examples in this section and apply the method of MLDM. We will then compare our results with known exact solutions.

3.1 Example 1

Let's examine a nonlinear ordinary differential equation with the given initial condition, where the prime denotes the differentiation with respect to x.

$$y' = -y + y^2, \quad y(0) = 2.$$

This particular nonlinear ODE has the exact solution: $y(x) = \dfrac{2}{e^x - 2}$, with a singularity at $x = \ln 2$. Next, applying the Laplace transform and using the given initial condition, we get,

$$\begin{aligned}\mathcal{L}[y'] &= -\mathcal{L}[y] + \mathcal{L}[y^2]\\ \Rightarrow sy(s) - y(0) &= -y(s) + \mathcal{L}[y^2]\\ \Rightarrow y(s) &= \frac{2}{s+1} - \frac{1}{s+1}\mathcal{L}[y^2].\end{aligned}$$

Now, applying the inverse Laplace transform to both sides, we get, $y(x) = 2e^{-x} - \mathcal{L}^{-1}\Big[\frac{1}{s+1}\mathcal{L}\big[y^2\big]\Big]$. In order to find the complete solution, we need to decompose the nonlinear part of the equation as an infinite sum as required by the superposition principle:

$$y = \sum_{n=0}^{\infty} y_n(x).$$

We then represent the nonlinear term as a series of Adomian polynomials as given: $y^2 = \sum_{n=0}^{\infty} A_n(y)$.

Applying this substitution in our equation we get: $\sum_{n=0}^{\infty} y_n(x) = 2e^{-x} - \mathcal{L}^{-1}\Big[\dfrac{1}{s+1}\mathcal{L}\big[\sum_{n=0}^{\infty} A_n(y)\big]\Big]$.

The recursive relation is as follows:

$$y_0 = 2e^{-x};\ y_1 = -\mathcal{L}^{-1}\Big[\frac{1}{s+1}\mathcal{L}\big[\sum_{n=0}^{\infty} A_0(y)\big]\Big]\ y_{n+1} = -\mathcal{L}^{-1}\Big[\frac{1}{s+1}\mathcal{L}\big[\sum_{n=0}^{\infty} A_n(y)\big]\Big], \quad n \geq 1.$$

In order to approximate the solution using only two terms, namely y_0, and y_1, we simplify our equation as follows:

$$y_1 = \mathcal{L}^{-1}\Big[\frac{-1}{s+1}\mathcal{L}\big[\sum_{n=0}^{\infty} A_0(y)\big]\Big] = \mathcal{L}^{-1}\Big[\frac{-1}{s+1}\mathcal{L}\big[y_0^2\big]\Big] = \mathcal{L}^{-1}\Big[\frac{-1}{s+1}\mathcal{L}\big[4e^{-2x}\big]\Big] = \mathcal{L}^{-1}\Big[\frac{-4}{(s+1)(s+2)}\Big] = 4e^{-2x}(1 - e^x).$$

In view of above modified recursive relation, combining our result obtained from y_0 and y_1, we achieve an analytical solution after simplifying: $y(x) = 2e^{-2x}(2 - e^x)$.

We can use the expressions for y_0 and y_1, to approximate the next iteration y_2:

$$\begin{aligned}
y_0 &= 2e^{-x},\\
y_1 &= -\mathcal{L}^{-1}\Big[\frac{1}{s+1}\mathcal{L}\big[\sum_{n=0}^{\infty} A_0(y)\big]\Big],\\
y_2 &= -\mathcal{L}^{-1}\Big[\frac{1}{s+1}\mathcal{L}\big[\sum_{n=0}^{\infty} A_1(y)\big]\Big],\\
y_{n+1} &= -\mathcal{L}^{-1}\Big[\frac{1}{s+1}\mathcal{L}\big[\sum_{n=0}^{\infty} A_n(y)\big]\Big], \quad n \geq 1.
\end{aligned}$$

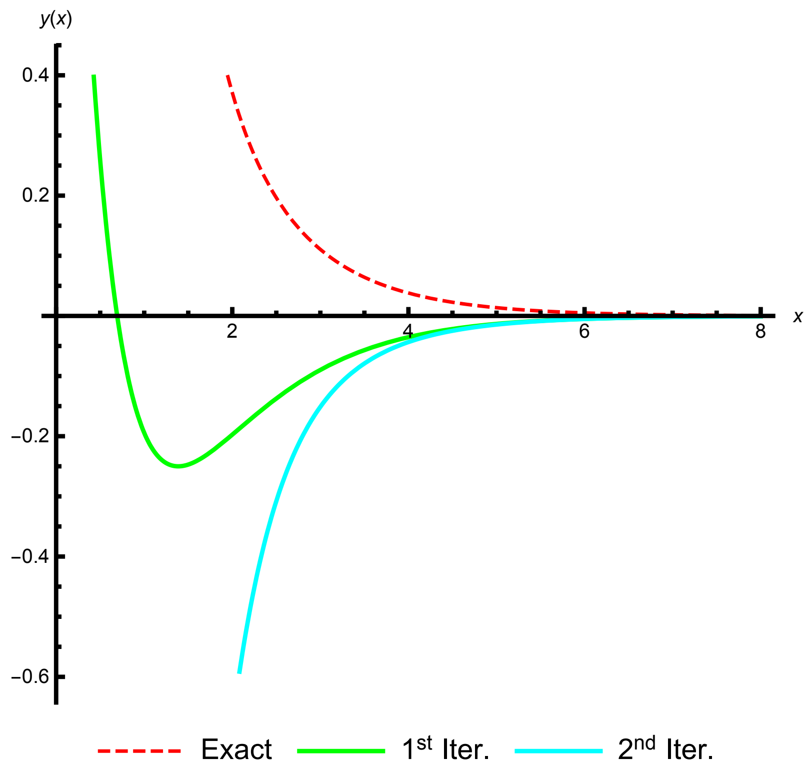

Figure 1: Figure presents the comparison between the exact solution (red line), the approximated solution with the 1^{st} iteration (green line), and the approximated solution with the 2^{nd} iteration (blue line) for the problem in example 1.

Further continuing the calculations, we obtain,

$$\begin{aligned}
y_2 &= -\mathcal{L}^{-1}\Big[\frac{1}{s+1}\mathcal{L}\big[2(2e^{-2x})\big(4[e^{-x} - e^{-2x}]\big)\big]\Big]\\
&= -\mathcal{L}^{-1}\Big[\frac{1}{s+1}\mathcal{L}\big[16\big(e^{-2x} - e^{-3x}\big)\big]\Big]\\
&= -16\mathcal{L}^{-1}\Big[\frac{1}{(s+1)(s+2)} - \frac{1}{(s+1)(s+3)}\Big]
\end{aligned}$$

$$= -16\mathcal{L}^{-1}\Big[-\frac{1}{s+2}+\frac{1}{s+3}\Big]$$
$$= -16\big(-e^{-2x}+e^{-3x}\big).$$

The new solution $y(x)$ can be obtained by adding the expression of y_2, with the expressions for y_0 and y_1 as follows: $y(x) = -2e^{-x} - 20e^{-2x} - 16e^{-3x}$.

3.2 Example 2

Next, we would like to apply MLDM to the NLPDE with boundary conditions given below:

$$\text{PDE: } u_{xx} = -u^2 + u_y^2, \qquad \text{BC: } u(0,y) = 0, u_x(0,y) = e^y.$$

By applying the MLDM to the nonlinear PDE, we get:

$$\mathcal{L}\Big[u_{xx}\Big] = \mathcal{L}\Big[-u^2 + u_y^2\Big]$$
$$\Rightarrow s^2u(s,y) - su(0,y) - u_x(0,y) = \mathcal{L}\Big[-u^2 + u_y^2\Big]$$
$$\Rightarrow u(s,y) = \frac{e^y}{s^2} - \frac{1}{s^2}\mathcal{L}\Big[u^2 - u_y^2\Big].$$

Applying the Laplace inverse on both sides, we get, $u(x,y) = xe^y - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}\big[uu - u_yu_y\big]\Big]$. By expressing $u(x,y)$ as an infinite series, the above expression for $u(x,y)$ can be recast as follows:

$$\sum_{n=0}^{\infty} u_n(x,y) = xe^y - \mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}\Big(\sum_{n=0}^{\infty}\big[A_n - B_n\big]\Big)\Big].$$

By assuming $u_0(x,y) = xe^y$, we can proceed to find u_1 next,

$$u_1(x,y) = -\mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}\Big(\sum_{n=0}^{\infty}\big[u_0^2 - u_{0y}^2\big]\Big)\Big] = -\mathcal{L}^{-1}\Big[\frac{1}{s^2}\mathcal{L}\Big(\sum_{n=0}^{\infty}\big[x^2e^{2y} - x^2e^{2y}\big]\Big)\Big] = 0,$$

which subsequently renders rest of the u's to be 0. Thus, the final solution is $u(x,y) = xe^y$. As stated in the introduction, a special case of nonlinear differential equation using MLDM will usually produce noise terms if the exact solution is equal to the first iteration u_0. These noise terms are canceled and the remaining terms do not contribute to the solution.

3.3 Example 3

In the next example, MLDM is applied to a coupled system of nonlinear PDE with initial conditions:

$$\text{PDE: } u_t - v_xw_y = 1; \quad v_t - w_xu_y = 5; \quad w_t - u_xv_y = 5,$$
$$\text{IC: } u(x,y,0) = x+2y; \quad v(x,y,0) = x-2y; \quad w(x,y,0) = -x+2y.$$

Applying the Laplace transform on each PDE, subsequently simplifying the expressions after applying the initial conditions, we arrive at:

$$\mathcal{L}[u_t] = \mathcal{L}[1] + \mathcal{L}[v_xw_y] \Rightarrow su(x,y,s) - u(x,y,0) = \frac{1}{s} + \mathcal{L}[v_xw_y] \Rightarrow u(x,y,s) = \frac{x+2y}{s} + \frac{1}{s^2} + \frac{1}{s}\mathcal{L}[v_xw_y]$$

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$\mathcal{L}[v_t] = \mathcal{L}[5]+\mathcal{L}[w_x u_y] \Rightarrow sv(x,y,s)-v(x,y,0) = \frac{5}{s}+\mathcal{L}[w_x u_y] \Rightarrow v(x,y,s) = \frac{x-2y}{s}+\frac{5}{s^2}+\frac{1}{s}\mathcal{L}[w_x u_y]$

$\mathcal{L}[w_t] = \mathcal{L}[5]+\mathcal{L}[u_x v_y] \Rightarrow sw(x,y,s)-w(x,y,0) = \frac{5}{s}+\mathcal{L}[u_x v_y] \Rightarrow w(x,y,s) = \frac{-x+2y}{s}+\frac{5}{s^2}+\frac{1}{s}\mathcal{L}[u_x v_y].$

Applying the inverse Laplace transform on both sides of each equation of the system, we obtain:

$$u(x,y,t) = x+2y+t+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[v_x w_y]\right],$$
$$v(x,y,t) = x-2y+5t+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[w_x u_y]\right],$$
$$w(x,y,t) = -x+2y+5t+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[u_x v_y]\right].$$

We represent each equation of the system as a series with A_n, B_n, and C_n representing Adomian polynomials respectively,

$$\sum_{n=0}^{\infty} u_n(x,y,t) = x+2y+t+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(\sum_{n=0}^{\infty} A_n(v,w)\Big)\right],$$
$$\sum_{n=0}^{\infty} v_n(x,y,t) = x-2y+5t+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(\sum_{n=0}^{\infty} B_n(w,u)\Big)\right],$$
$$\sum_{n=0}^{\infty} w_n(x,y,t) = -x+2y+5t+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(\sum_{n=0}^{\infty} C_n(u,v)\Big)\right].$$

The recursive relation for the system follows as below:

$u_0(x,y,t) = x+2y;\ u_1(x,y,t) = t+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(\sum_{n=0}^{\infty} A_0(v,w)\Big)\right];\ u_{n+1}(x,y,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(\sum_{n=0}^{\infty} A_n(v,w)\Big)\right],$

$v_0(x,y,t) = x-2y;\ v_1(x,y,t) = 5t+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(\sum_{n=0}^{\infty} B_0(w,u)\Big)\right];\ v_{n+1}(x,y,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(\sum_{n=0}^{\infty} B_n(w,u)\Big)\right],$

$w_0(x,y,t) = -x+2y;\ w_1(x,y,t) = 5t+\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(\sum_{n=0}^{\infty} C_0(u,v)\Big)\right];\ w_{n+1}(x,y,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\Big(\sum_{n=0}^{\infty} C_n(u,v)\Big)\right].$

By carrying out some simple algebra, we recover the solution for u, v, and w.

$$u_1(x,y,t) = t+\mathcal{L}^{-1}\frac{1}{s}\mathcal{L}[v_{0x}w_{0x}] \Rightarrow t+\mathcal{L}^{-1}\frac{1}{s}\mathcal{L}[2] \Rightarrow t+\mathcal{L}^{-1}\left[\frac{2}{s^2}\right] \Rightarrow t+2t \Rightarrow 3t,$$

which consequently yields the complete solution for $u(x,y,t) = u_0(x,y,t)+u_1(x,y,t) = x+2y+3t$. Similarly, we can obtain the solutions for v and w as well:

$v(x,y,t) = x-2y+3t$, and $w(x,y,t) = -x+2y+3t$.

4 Conclusion

In this paper, we demonstrated the efficiency of the modified Laplace decomposition method as well as the special cases involved while analyzing its application. We solved three nonlinear differential equations with initial conditions. In the first example, after solving the equation using the MLDM, we compared first two iterations graphically and demonstrated how different these interpolations were compared with the exact solution. In example two, it was validated with proper justification that certain terms (also known as noise terms) which originate while seeking the solutions, cancel each other out to yield the exact solution. In example three, the method was applied successfully to a system of nonlinear partial differential equations to attain the exact solution. This technique has proved to be a powerful tool while tackling these nonlinear differential equations which model a multitude of real-world phenomena observed in the field of applied physics, engineering, and other sciences.

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