NEW SCHWARZ NORMS ON B(H)

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Abstract
In this paper, we give results on new Schwarz norms in B(H). We also give a characterization for a new class of norms in Banach space setting.

1 Introduction

A norm $\|.\|^{*}$ on $B(H)$ which is equivalent to the operator norm $\|.\|$ is called a Schwarz norm if $\|T\| \leq 1$ implies $\|f(T)\| \leq \|F\|_{\infty} \equiv \max_{|z|\leq 1}|f(z)|$ ……………………..(*) for any analytic function f with f(0)= 0 and $\|F\|_{\infty} < 1$. Von Neumann [11] first showed that if $T \in B(H)$ then the usual operator norm $\|T\| = \sup\{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$ is a Schwarz norm using the spectral representation of aa unitary operator U i.e $f(U) = \int_0^{2\Pi} f(e^{i\theta})dE(\theta)$ generates a norm $\|f(U)x\|^2 = \int_0^{2\Pi}|f(e^{i\theta})|^2 dE\|\theta\|^2$ where $E(\theta)$ is a positive spectral measure of U. the inequality (*) above then follow from this norm. now the numerical radius of an operator $T \in B(H)$ is defined as $w(T) = \sup\{|z| : z \in W(T)\}$ where W(T) is the numerical range of T, i.e the set W(T)= $\{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$. Berger and Stampfli [2] proved that the numerical radius w(T) is a Schwartz norm using the theory of unitary dilation i.e w(T) ≤1 if and only if there is a unitary operator U on $K \supset H$ such that $T^n = 2PU^n / H(n = 1,2,...)$. Nagy and Foias [3] and later other papers improved on this to obtain the ρ - radius, $w_p(T)$ of an operator as $W_p(T) \equiv \inf\{\lambda > 0; \frac{1}{\lambda}T \in C_p\}$ where C_p is the class of operator with ρ dilations. Thus for a complex valued function f(z) defined and analytic on the closed unit disk with f(0) = 0, if T has a ρ dilation U, then by series expansion, $f(T)^n = \rho Pf(U)^n / H(n = 1,2,...)$ and it can then be proved that $w_p(f(T)) \leq \|f\|_{\infty}$ so that the inequality (*)is achieved .

Using the two norms $\|T\|$ and w(T) (as proved by Von Neumann and Berger –Stampfli to be Schwartz norms), William [1] constructed a class S_c of operators which he used to build a family of Schwartz norms.

2. Preliminaries

We will in this section give the definitions that will be essential in our study. In the following **K=R or C**

Definition 2.1 if $T \in B(H)$, then the operator $T^* : H \to H$ defined by $\langle Tx, y\rangle = \langle x, T^* y\rangle \ \forall x, y \in H$ is called the adjoint of T. (T^* is also in B(H) and $\| T^* \| = \| T^* \|$

Definition 2.2 an operator $T \in B(H)$ is said to be self adjoint if $T^*=T$ and if T is linear on a linear subspace M of Hilbert space H into M then it is said to be Hermitian if in addition $\langle Tx, y\rangle = \langle x, Ty\rangle \forall x, y \in M$

Definition 2.3 Let H be a complex Hilbert space and $T \in B(H)$. Then there exists unique self adjoint operators A,B $\in B(H)$ such that $T = A + iB$, A and B are given by $A = \frac{1}{2}(T + T^*), B = \frac{1}{2i}(T + T^*)$ so that A is called real part of T denoted by ReT and B the imaginary part of T denoted by ImT. Note that $\operatorname{Re}\langle Tx, x\rangle = \langle (\operatorname{Re}T)x, x\rangle$ for every x $x \in H$, indeed $\langle Tx, x\rangle = \frac{1}{2}\langle (T + T^*)x, x\rangle + i\frac{1}{2}\left\langle \left(\frac{TT^*}{2}\right)x, x\right\rangle$ and $\langle Tx, x\rangle$ being a complex number we have $\langle Tx, x\rangle$=a+ib, where a,b are real numbers given by $a = \langle (\operatorname{Re}T)x, x\rangle, b = \langle (\operatorname{Im}T)x, x\rangle$

Definition 2.4 let H be a complex Hilbert space and T$\in B(H)$, the numerical range of T is the set W(T)l$\subset C$ defined by $W(T) = \{\langle Tx, x\rangle : x \in H, and, \|x\| = 1\}$

Definition 2.5 the numerical radius w(T) of an operator $T \in B(H)$ is the number defined by the relation $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$

Definition 2.6 let X,Y be normed liner spaces over K and $T : X \to Y$ be a linear transformation , then T is said to be compact if for every bounded subset M of X, the image $\overline{T(M)}$ (strongly closure of T(M) in X) is compact or equivalently , if X,Y be normed linear spaces over K and $T : X \to Y$ be a linear T is said to be compact if and only if for every bounded sequence (X_n) of elements of X, the sequence $(T(X_n))$ has a subsequence which converges strongly in Y. the set K(X,Y) of all compact linear operators $T : X \to Y$ is a linear subspace of B(X,Y) which is a set of all bounded linear operators $T : X \to Y$

Definition 2.7 a Banach algebra **B** is a Banach space $(\mathbf{B}, \|.\|)$ in which for every x,y $\in$ **B** such that

i. (λx)y=λ(xy)=x(λy) for all λ in **K**

ii. (x + y)z = xz + yz for all x,y,z in **B**
iii. x(y+ z) = xy + xz for all x,y,z in **B**
iv. $\|xy\| \le \|x\|\|y\|$ x,y,z in **B**

Definition 2.8 suppose A is arbitrary Banach algebra (commutative or not), a mapping $*: A \to$ **A** is called an involution of **A** or **A** is called an involutive Banach space if;

1. $(x+y)^* = x^* + y^*$
2. $(\lambda x)^* = \bar{\lambda} x^* \lambda \in$ **C**
3. $(\lambda x)^* = y^* x^*$
4. $(x^*)^* = x$ for all x,y$\in$ **A**

An involutive Banach algebra **A** is called a **B*** algebra if $\|x^* x\| = \|x\|^2$ for all x$\in$ **A**

Definition 2.9 let X be a linear space over K and M be a linear subspace of X. for each x$\in X$ we define x+M=$\{x+y : y \in M\}$, and if $x, x' \in X$ then x + M= x' +M if and only if x, $x' \in M$

Definition let $(X, \|.\|)$ be a normed linear space and M be a closed linear subspace of X ,for each element x+M in X/M, define a function $\|x+M\| = \inf\{\|x+y\| : y \in M\} = dis(x, M)$ then $\|.\|$ is a norm in X/M i.e (X/M, $\|.\|$) is a Banach space if (X/M, $\|.\|$)is a Banach space. If M is not closed then $\|x+M\| = 0 \Rightarrow x \in M$ and $\therefore x + M \ne M$, the zero element of X/M. therefore $\|.\|$ is a seminorm.

Definition suppose X in the above definition is **B(H)**;then **B(H)/K(H)=** $\{T + K(H) : T \in B(H)\}$ is called a Calkin algebra. For each Tin K(H), there corresponds a unique in $\hat{T}$ on **B(H)/K(H)** and this correspondence given by $T \mapsto \hat{T}$ and can also be given by $T \to (T + K(H)) = \hat{T}$

Main results

Proposition If $\|T\|_C$ is a norm and $\|\hat{T}\|_C$ is a seminorm, then the sum is a Schwarz norm i.e taking the sum of two different Schwarz norm applied to T and to the image of T in the Calkin algebra. For any c$\ge$1 we define on B(H) the function $\|T\|^*_C = \|T\|_C + \|\hat{T}\|_C$ $\forall$T $\in$ B(H) where $\hat{T}$ denotes the image of T in the Calkin algebra and $\|\hat{T}\|_C$ being a seminorm as indicated in definition 1.2.19. then T$\to$ $\|T\|^*_C$ is a Schwarz norm on B(H) and is not in the class constructed by Williams.

proof. First we remark that we can construct a more general Schwarz norm on B(H) by taking the sum of two different Schwarz norms applied to T and to the image of T in the Calkin algebra.Also since ‖T‖$_C$ is a norm and ‖ $\hat{T}$ ‖$_C$is a seminorm, it follows that the sum is a Schwarz norm. Suppose that *Q* is a positive hermitian operator with the property $0 < mI \le Q \le MI$,

where $m = \inf\{\langle Tx, x\rangle : \|x\| = 1\}$ $M = \sup\{\langle Tx, x\rangle : \|x\| = 1\}$ Then we can construct the operator $Q^{\frac{1}{2}}$ which is also positive and invertible. The following new class S_Q of operators is a generalization of the class S_C to which it reduces when $Q = cI$

Definition. If Q is a Hermitian operator 0 < mI < Q < MI then the class S_Q is the set of all operators T ∈ B(H) with the following properties

1. δ(T) is in the unit disk.
2. $\mathrm{Re}\left(I + \sum Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} z^n \right) \ge 0$, for all |z|<1

We can prove some results about this class as for the class S_C obtained by Williams.

Theorem 2.15. If f is a rational function with no poles in the closed unit disk and ‖f‖$_\infty$ < 1,f(0) = 0 then for any T ∈ S_Q, f(T) ∈ S_Q In this proof, we use the approach of Williams [1]:

Proof:

The function $z \mapsto \left\langle \left(\sum_{n=1}^{\infty} Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} z^n \right) x, x \right\rangle$ is with real part positive. By the Herglotz theorem ,there exists a positive measure μ_x such that

$\|x\|^2 + c\sum_{n=1}^{\infty} z^n \left\langle Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} x, x \right\rangle = \int_0^{2\Pi} d\mu_x(t)$for all |z|<1 now,

From these relations ,we obtain immediately that for any polynomial *p*(*z*) = $\sum a_i z^i$ and any $x \in H$, $P\left\langle \left(Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} \right) x, x \right\rangle = 2\int_0^{2\Pi} p(e^{it}) d\mu_x(t)$ and if we take $p^n(z)$,we obtain $P^n\left\langle \left(Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} \right) x, x \right\rangle = 2\int_0^{2\Pi} p^n(e^{it}) d\mu_x(t)$ This implies that if $\|p\|_\infty = 1$, $p^n(Q^2\ T\ Q^2)$ is a bounded operator and for *z*, |*z*|< 1, we obtain.

$$\left\langle 1 + c\sum_{n=1}^{\infty} z^n p^n \left(Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} \right) x, x \right\rangle = \|x\|^2 + 2\sum_{n=1}^{\infty} z^n \int_0^{2\Pi} p^n(e^{it}) d\mu_x(t) =$$

$\int_0^{2\Pi} \frac{1 + zp(e^{it})}{1zp(e^{it})} d\mu_x(t)$From these relations , we obtain immediately that for any polynomial *p*(T) $\in S_Q$ when p is a polynomial . now if f is any functional which is rational and with no poles in the closed unit disk, then f(T) $\in S_Q$. Now this theorem shows that S_Q is a family of distinct Schawrz norms. f(T) $\in S_Q$

Proposition 2.16. The operator T ∈ B(H) is in S_Q if and only if :

1. δ (T)is in the unit disk
2. $\mathrm{Re}\left\langle\left(Q^{\frac{1}{2}}(I\ \ zT)^{1}Q^{\frac{1}{2}}x, x\right)\right\rangle \langle Qx, x\rangle + \|x\|^2 \geq 0$

Proof; the condition,
$\mathrm{Re}\left(I + \sum Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} z^n \geq 0 \right)$ is equivalent to the following Re[($Q^{1/2}$(I zT)^1$Q^{1/2}$ Q+I)x,x>] Which is our assertion. From this characterization we obtain the following result

Proposition 2.17. If Q ≥ 1, then T ∈ S_Q if and only if

1. δ (T) is in the unit disk
2. Re< $Q^{1/2}$ (I zT) $Q^{1/2}$x,x>‖ $Q^{1/2}$x‖^2 ‖x‖^2=<(Q I)x, x>

Proof:
This follows directly from the above proposition 3.1.4. The following theorem gives information about the S_Q class which is similar to that given in proposition 2 for the S_c class.
Proposition 2.18. If Q is a positive hermitian operator ,then the following assertions hold.

1. $S_Q = S_Q^* = \{T^* : T \in S_Q\}$
2. If $Q_1 < Q_2$ then $S_{Q_2} \subseteq S_{Q_1}$
3. For Q ≥ I, S_Q is a convex bounded ,circled and weakly compact set in (H) (it is also in the neighborhood of zero)

Proof: Now we prove the assertion (1) above, Since (T) ⊂ U, it follows that δ (T *) ⊂ U. Indeed δ (T*) = (δ (T))* (the star on the right side denotes the complex conjugation , i.e, (δ (T))* = {z* : z ∈ (T)}. Moreover ,since |z| = |z*| < 1,for all x ∈ H

Thus T * ∈ S_Q, i.e $S_Q^* \subset S_Q$,where $S_c^* = \{T^* : T \in S_c\}$. Likewise $S_Q \subset S_Q^*$ and hence $S_Q = S_Q^*$. To prove (2): let $Q_2 < Q_1$.Now T ∈ S_{Q_1} ⇒ (T) ⊂ U and
(Q_1 1) ‖T x‖^2 + |2 Q_1^1 ‖T x, x| ≤‖x‖^2
⇒ (Q_2 1) ‖T x‖^2 +|2 Q_2^1‖Tx, x|≤‖x‖^2.
Thus $T \in S_Q$. Hence $S_{Q_1} \subseteq S_{Q_2}$. To prove the convexity of S_c for c ≥ 1 ,we use the property (iv). If T_1 and T_2 are two operators and Q_2, Q_2 are their corresponding positive Hermitian operator as described just after proposition 3.1.1,then from

$\|T_1 + T_2\|^2 \leq 2(\|T_1\|^2 + \|T_2\|^2)$. Indeed $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$. Also $(\|T_1\| - \|T_2\|)^2 \geq 0 \Rightarrow \|T_1\|^2 + \|T_2\|^2 \geq 2\|T_1\|\,\|T_2\|$ thus $\|T_1x + T_2x\|^2 \leq \|T_1x\|^2 + \|T_2x\|^2 + 2\|T_1x\|\,\|T_2x\| \leq 2(\|T_1x\|^2 + \|T_2x\|^2)$. Now if T_1 and T_2 are members of S_Q ,then using condition (2) in proposition 3.1.5, and a simple calculation, we

have $1/2(T_1 + T_2) \in S_Q$. From the properties of S_Q in the proposition 3.1.6,we further obtain the following useful proposition.
Proposition 2.19. For any bounded hermitian operator Q > I, the function, $T \to T\|\|_Q = \inf\{s : T \in sS_Q\}$ is a Schwarz norm on B(H).From this class of Schwarz norms, we can obtain ,using the Calkin algebra, another class of Schwarz norms.
Proposition 2.20. Let Q_1 Q_2 be two bounded hermitian operators and $Q_i \geq I$ i = 1,2. In this case the function on B(H) defined by $T \mapsto \|T\|_{Q1} + \|\hat{T}\|_{Q2}$ where $\hat{T}$ denotes the image of T in the Calkin algebra of H, is a Schwarz norm on B(H)
Remark 2.21. The above construction of Schwarz norms can be given in the case of B^*-algebras. For the construction of Schwarz norms we can use the representations of the B^*-algebra in the algebra B(H) for some H

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