# BL-algebras with pseudo MV-valuations

Yongwei Yang*, Yongzhao Wang, Qiaoling Wang

*School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China*

**Abstract**

The concept of pseudo MV-valuations on BL-algebras is introduced, which provides a new idea for the study of BL-algebras from MV-algebras. Some related characterizations of pseudo MV-valuations and conditions for a function to be a pseudo MV-valuation are provided. The notion of pseudo MV-metric spaces is also given, and properties of pseudo MV-metric spaces are given by discussing pseudo MV-metrics induced by pseudo MV-valuations.

*Keywords:* BL-agelbras; pseudo MV-valuations; pseudo MV-metric spaces; pseudo MV-metrics

## 1. Introduction

Non-classical logic systems which lay logical foundation for dealing with uncertain information processing and fuzzy information, are uniquely determined by the algebraic properties of the structure of their truth values. BL-algebras as the algebraic structures for Hájek's basic logic were raised from the continuous $t$-norm, familiar in the fuzzy logic framework [1]. A particular subclass of BL-algebras are MV-algebras, the algebraic counterpart of Lukasiewicz infinite valued logic, and Turunen also pointed out that RS-BL-algebras are MV-algebras [2]. Other subclasses of BL-algebras are Product algebras and Godel algebras corresponding to Product fuzzy logic and Godel logic, respectively. The study of BL-algebras has experienced a tremendous growth over the recent years and the main focus has been on filters and ideals [3, 4, 5]

The notion of pseudo-valuation is first introduced into Hilbert algebra by Buşneag in [6], then he applied the notions of pseudo-valuations (valuations) into residuated lattices, and proved some theorems of extension for pseudo-valuations [7]. Following the research of Jun et al. [8], [9] investigated related characterizations of (implicative) pseudo-valuations on $R_0$-algebras, and showed that a pseudo-valuation on $R_0$-algebras is Boolean if and only if it is implicative. Yang and Xin introduced the notion of (positive implicative, implicative) pseudo pre-valuations and strong pseudo pre-valuations, and by using a congruence relation induced via a pseudo valuation, they constructed a quotient structure and proved certain isomorphism theorems [10]. [11] gave the notion of pseudo valuations on hoop-algebras, and investigated the relationship between pseudo-valuations and filers.

*Corresponding author. yangyw2010@126.com (Y. W. Yang).

Lele and Nganou [3] claimed that most results from the theory of MV-algebras remain unchanged in BL-algebras,though their proofs are usually quite different in the BL-algebras settings. And we notice that the notion of pseudo-valuations is a real-valued function, so why not discuss BL-algebras by using the structural properties of MV-algebras? Therefore we introduce the notion of pseudo MV-valuations by a function from a BL-algebra to an MV-algebra. Some characterizations of pseudo MV-valuations and conditions for a function to be a pseudo MV-valuation are provided. Some properties of pseudo MV-metric spaces and pseudo MV-metrics induced by pseudo MV-valuations are also discussed.

## 2. Preliminaries

In this section, we give the basic definitions and results of EQ-algebras that are useful for subsequent discussions.

**Definition 2.1.** [1] *An algebra* $(L, \wedge, \vee, \odot, \to, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *is called a BL-algebra if it satisfies the following conditions: for all* $x, y, z \in L$,

(BL-1) $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
(BL-2) $(L, \odot, 1)$ *is a commutative monoid,*
(BL-3) $x \odot y \leq z$ *if and only if* $x \leq y \to z$,
(BL-4) $x \odot (x \to y) = x \wedge y$,
(BL-5) $(x \to y) \vee (y \to x) = 1$.

Let $L$ be a BL-algebra. If $x \vee \bar{x} = 1$ for any $x \in L$, then $L$ is called a Boolean algebra, where $\bar{x} = x \to 0$; if $L$ satisfies the double negation, i.e., $\bar{\bar{x}} = x$ for any $x \in L$, then $L$ is called an MV-algebra.

**Lemma 2.2.** [1, 3] *In any BL-algebra* $(L, \wedge, \vee, \odot, \to, 0, 1)$, *the following relations hold: for any* $x, y, z \in L$,

(1) $x \leq y$ *if and only if* $x \to y = 1$, $x \odot y = 0$ *if and only if* $x \leq \bar{y}$;
(2) $x \odot (x \to y) \leq y$, $x \odot y \leq x \wedge y \leq x \to y$, $x \leq y \to x$;
(3) $x \to y \leq (y \to z) \to (x \to z)$, $y \to x \leq (z \to y) \to (z \to x)$;
(4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;
(5) $\bar{0} = 1$, $\bar{1} = 0$, $1 \to x = x$, $x \to 1 = 1$;
(6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$, $(x \wedge y) \to z = (x \to z) \vee (y \to z)$;
(7) *if* $x \leq y$, *then* $y \to z \leq x \to z$, $z \to x \leq z \to y$, $x \odot z \leq y \odot z$;
(8) $x \to y = x \to (x \wedge y)$, $x \to y \leq (x \wedge z) \to (y \wedge z)$, $x \to y \leq (x \odot z) \to (y \odot z)$.

**Definition 2.3.** [1] *Let* $X$, $Y$ *be BL-algebras. A function* $f : X \to Y$ *satisfying* $f(1_X) = 1_Y$ *and* $f(0_X) = 0_Y$ *is called a homomorphism if* $f(a * b) = f(a) \star f(b)$, *where* $* \in \{\wedge_X, \vee_X, \odot_X, \to_X\}$ *in* $X$ *and* $\star \in \{\wedge_Y, \vee_Y, \odot_Y, \to_Y\}$ *in* $Y$.

An algebra $(M, \oplus, \neg, 0)$ of type $(2, 2, 0)$ is called an MV-algebra if it satisfies the following axioms: for any $x, y, z \in M$,

(MV-1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(MV-2) $x \oplus y = y \oplus x$,
(MV-3) $x \oplus 0 = x$,
(MV-4) $\neg\neg x = x$,
(MV-5) $x \oplus \neg 0 = \neg 0$,
(MV-6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

Let $(M, \oplus, \neg, 0)$ be an MV-algebra, for any $x, y \in M$, we put $1 = \neg 0$, $x \otimes y = \neg(\neg x \oplus \neg y)$, $x \to y = \neg x \oplus y$, $x \ominus y = x \otimes \neg y$, $x \vee y = \neg(\neg x \oplus y) \oplus y = (x \ominus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$.

**Proposition 2.4.** ([12]) *Let $(M, \oplus, \neg, 0)$ be an MV-algebra. Then the following assertions are valid: for any $x, y, z, s, t \in M$,*

(1) *$x \leq y$ if and only if $\neg x \oplus y = 1$ if and only if $x \ominus y = 0$;*
(2) *$x \ominus y \leq z$ if and only if $x \leq y \oplus z$;*
(3) *$x \otimes \neg x = 0$, $x \oplus \neg x = 1$, $x \otimes y = y \otimes x$;*
(4) *$x \otimes y \leq x \wedge y \leq x \vee y \leq x \oplus y$;*
(5) *$x \ominus z \leq (x \ominus y) \oplus (y \ominus z)$;*

## 3. Pseudo MV-valuations on BL-algebras

In the section, we introduce the notion of pseudo MV-valuations, and give some characterizations of a pseudo MV-valuation on BL-algebras. In what follows, unless mentioned otherwise, $L$ is a BL-algebra and $M$ is an MV-algebra.

**Definition 3.1.** *Let $\varphi : L \to M$ be a function from a BL-algebra $L$ to an MV-algebra $M$. Then $\varphi$ is called a pseudo MV-valuation on $L$ if it satisfies the following conditions: for any $x, y \in L$,*

(1) *$\varphi(1) = 0$,*
(2) *$\varphi(y) \leq \varphi(x) \oplus \varphi(x \to y)$.*

A pseudo MV-valuation $\varphi$ is called an MV-valuation if $\varphi(x) = 0$ implies $x = 1$.

For the sake of simplicity and better understanding, we give the following example.

**Example 3.2.** *Let $L = \{0, a, b, 1\}$, where $0 < a < 1$ and $0 < b < 1$. Define $\odot$ and $\to$ as follows:*

| $\odot$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a |
| b | 0 | 0 | b | b |
| 1 | 0 | a | b | 1 |

| $\to$ | 0 | a | b | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| a | b | 1 | b | 1 |
| b | a | a | 1 | 1 |
| 1 | 0 | a | b | 1 |

*and $\wedge$ and $\vee$ operations on $L$ as min and max, respectively. Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra.*

*Let $E = \{a, b, c\}$ and $\mathscr{P}(E)$ be the power set of $E$. Let $\oplus$, $\neg$ and $0$ denote, respectively, the join, the complement and the smallest element in $M := \mathscr{P}(E)$. It is clear that $(M, \oplus, \neg, 0)$ is an MV-algebra.*

*Define $\varphi : L \rightarrow M$ by $\varphi(0) = \{b, c\}$, $\varphi(a) = \{c\}$, $\varphi(b) = \{b\}$, $\varphi(1) = \emptyset$. Routine calculation shows that $\varphi$ is a pseudo MV-valuation.*

**Proposition 3.3.** *Let $\varphi : L \rightarrow M$ be a pseudo MV-valuation on $L$, then the following properties hold: for any $x, y, w, v \in L$,*

(1) *$x \leq y$ implies that $\varphi(y) \leq \varphi(x)$;*
(2) *$0 \leq \varphi(x)$;*
(3) *$\varphi(x \rightarrow z) \leq \varphi(x \rightarrow y) \oplus \varphi(y \rightarrow z)$;*
(4) *$\varphi((x \rightarrow y) \rightarrow z) \leq \varphi(x \rightarrow (y \rightarrow z))$;*
(5) *$\varphi((x \wedge w) \rightarrow (y \wedge v)) \leq \varphi(x \rightarrow y) \oplus \varphi(w \rightarrow v)$;*
(6) *$\varphi((x \odot w) \rightarrow (y \odot v)) \leq \varphi(x \rightarrow y) \oplus \varphi(w \rightarrow v)$.*
(7) *$\varphi((x \odot z) \rightarrow (y \odot z)) \leq \varphi(x \rightarrow y)$.*

**Proof.** (1) Let $x, y \in L$ such that $x \leq y$, then we have $x \rightarrow y = 1$, and so $\varphi(y) \leq \varphi(x) \oplus \varphi(x \rightarrow y) = \varphi(x) \oplus \varphi(1) = \varphi(x) \oplus 0 = \varphi(x)$.

(2) Since $x \leq 1$ for any $x \in L$, according to (1), we get $0 = \varphi(1) \leq \varphi(x)$.

(3) Consider that $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$, we get that $\varphi(x \rightarrow z) \leq \varphi((y \rightarrow z) \rightarrow (x \rightarrow z)) \oplus \varphi(y \rightarrow z) \leq \varphi(x \rightarrow y) \oplus \varphi(y \rightarrow z)$.

(4) Since $x \odot y \leq y \leq x \rightarrow y$, then $(x \rightarrow y) \rightarrow z \leq (x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ by Lemma 2.2, and therefore $\varphi(x \rightarrow (y \rightarrow z)) \leq \varphi((x \rightarrow y) \rightarrow z)$.

(5) For any $x, y, w, v \in L$, we have $x \rightarrow y \leq (x \wedge w) \rightarrow (y \wedge w)$ and $w \rightarrow v \leq (y \wedge w) \rightarrow (y \wedge v)$ by Lemma 2.2. By (1) and (3), we obtain that $\varphi(x \rightarrow y) \oplus \varphi(w \rightarrow v) \geq \varphi((x \wedge w) \rightarrow (y \wedge v)) \oplus \varphi((y \wedge w) \rightarrow (y \wedge v)) \geq \varphi((x \wedge w) \rightarrow (y \wedge v))$.

(6) The proof is similar to that of (5).

(7) We put $w = z$ and $v = z$ in (6), then $\varphi((x \odot z) \rightarrow (y \odot z)) \leq \varphi(x \rightarrow y) \oplus \varphi(z \rightarrow z) = \varphi(x \rightarrow y)$.

In the following, some conditions for a function $\varphi : L \rightarrow M$ to be a pseudo MV-valuation will be given.

**Theorem 3.4.** *Let $\varphi : L \rightarrow M$ be a function such that $\varphi(1) = 0$. Then $\varphi$ is a pseudo MV-valuation if and only if $x \leq y \rightarrow z$ implies $\varphi(z) \leq \varphi(x) \oplus \varphi(y)$ for any $x, y, z \in L$.*

**Proof.** Suppose that $\varphi$ is a pseudo MV-valuation and $x \leq y \rightarrow z$. It follows that $\varphi(y \rightarrow z) \leq \varphi(x)$, and so $\varphi(z) \leq \varphi(y) \oplus \varphi(y \rightarrow z) \leq \varphi(x) \oplus \varphi(y)$.

Conversely, since $x \rightarrow y \leq x \rightarrow y$, thus $\varphi(y) \leq \varphi(x \rightarrow y) \oplus \varphi(x)$, and so $\varphi$ is a pseudo MV-valuation.

The following results are immediately from Theorem 3.4.

**Proposition 3.5.** *Let $\varphi : L \to M$ be a pseudo MV-valuation on $L$. Then for any $x, y, w, v \in L$, we have:*

(1) $(\varphi(x) \ominus \varphi(y)) \vee (\varphi(y) \ominus \varphi(x)) \leq \varphi(x \odot y) \leq \varphi(x) \oplus \varphi(y)$;
(2) $(\varphi(x) \ominus \varphi(y) \vee \varphi(y) \ominus \varphi(x)) \leq \varphi(x \wedge y) \leq \varphi(x) \oplus \varphi(y)$;
(3) $\varphi((x \to w) \to (y \to v)) \leq \varphi(y \to x) \oplus \varphi(w \to v)$.

**Proof.** (1) For any $x, y \in L$, $x \odot y \leq x \odot y$, then $x \leq y \to (x \odot y)$, and so $\varphi(x \odot y) \leq \varphi(x) \oplus \varphi(y)$ by Theorem 3.4. As for the inverse inequality, from $x \odot y \leq x \to y$ and $y \odot x \leq y \to x$, we obtain that $\varphi(y) \leq \varphi(x \odot y) \oplus \varphi(x)$ and $\varphi(x) \leq \varphi(x \odot y) \oplus \varphi(y)$, that is, $\varphi(y) \ominus \varphi(x) \leq \varphi(x \odot y)$ and $\varphi(x) \ominus \varphi(y) \leq \varphi(x \odot y)$, hence (1) holds.

(2) The proof is similar to that of (1).

(3) According to Proposition 3.3 (3), for any $x, y, w, v \in L$, we have that $\varphi((x \to w) \to (y \to v)) \leq \varphi((x \to w) \to (y \to w)) \oplus \varphi((y \to w) \to (y \to v)) \leq \varphi(y \to x) \oplus \varphi(w \to v)$.

**Definition 3.6.** *Let $(L, d)$ be an ordered pair, where $d : L \times L \to M$ is a function. If $d$ satisfies the following conditions: for any $x, y, z \in L$,*

(1) $d(x, x) = 0$,
(2) $d(x, y) = d(y, x)$,
(3) $d(x, z) \leq d(x, y) \oplus d(y, z)$,

*then $(L, d)$ is called a pseudo MV-metric space. Moreover, if $d(x, y) = 0$ implies $x = y$, then $(L, d)$ is called an MV-metric space.*

**Theorem 3.7.** *Let $\varphi : L \to M$ be a pseudo MV-valuation. Define a function $d_\varphi : L \times L \to M$ by $d_\varphi(x, y) = \varphi(x \to y) \oplus \varphi(y \to x)$ for any $x, y \in L$, then $(L, d_\varphi)$ is a pseudo MV-metric space, where $d_\varphi$ is called the pseudo MV-metric induced by the pseudo MV-valuation $\varphi$.*

**Proof.** Obviously, $d_\varphi(x, y) \geq 0$, $d_\varphi(x, x) = 0$ and $d_\varphi(x, y) = d_\varphi(y, x)$ for any $x, y \in L$. Using Proposition 3.3 (3), we get that $d_\varphi(x, y) \oplus d_\varphi(y, z) = (\varphi(x \to y) \oplus \varphi(y \to x)) \oplus (\varphi(y \to z) \oplus \varphi(z \to y)) = (\varphi(x \to y) \oplus \varphi(y \to z)) \oplus (\varphi(z \to y) \oplus \varphi(y \to x)) \geq \varphi(x \to z) \oplus \varphi(z \to x) \geq d_\varphi(x, z)$. Hence $(M, d_\varphi)$ is a pseudo MV-metric space.

**Proposition 3.8.** *Let $\varphi : L \to M$ be a pseudo MV-valuation and $d_\varphi$ the pseudo MV-metric induced by $\varphi$. Then the following inequalities are valid: for any $x, y, z, w, v \in L$,*

(1) $\max\{d_\varphi(x \to z, y \to z), d_\varphi(z \to x, z \to y)\} \leq d_\varphi(x, y)$,
(2) $d_\varphi(x \to y, w \to v) \leq d_\varphi(x, w) \oplus d_\varphi(y, v)$,
(3) $d_\varphi(x \wedge z, y \wedge z) \leq d_\varphi(x, y)$,
(4) $d_\varphi(x \odot z, y \odot z) \leq d_\varphi(x, y)$,
(5) $d_\varphi(x \vee z, y \vee z) \leq d_\varphi(x, y)$,

(6) $(\varphi(x) \ominus \varphi(y)) \oplus (\varphi(y) \ominus \varphi(x)) \leq d_\varphi(x, y)$.

**Proof**. (1) For any $x, y, z \in L$, $y \to x \leq (x \to z) \to (y \to z)$ and $x \to y \leq (y \to z) \to (x \to z)$, then $\varphi(y \to x) \geq \varphi((x \to z) \to (y \to z))$ and $\varphi(x \to y) \geq \varphi((y \to z) \to (x \to z))$. It follows that $d_\varphi(x, y) = \varphi(y \to x) \oplus \varphi(x \to y) \geq \varphi((x \to z) \to (y \to z)) \oplus \varphi((y \to z) \to (x \to z)) = d_\varphi(x \to z, y \to z)$. Analogously, $d_\varphi(x, y) \geq d_\varphi(z \to x, z \to y)$. Hence $\max\{d_\varphi(x \to z, y \to z), d_\varphi(z \to x, z \to y)\} \leq d_\varphi(x, y)$.

(2) Since $d_\varphi$ is the pseudo-metric induced by $\varphi$, using (1) and Definition 3.6 (3), we find that $d_\varphi(x \to y, w \to v) \leq d_\varphi(x \to y, w \to y) \oplus d_\varphi(w \to y, w \to v) \leq d_\varphi(x, w) \oplus d_\varphi(y, v)$.

(3) For any $x, y, z \in L$, we obtain $d_\varphi(x \wedge z, y \wedge z) = \varphi((x \wedge z) \to (y \wedge z)) \oplus \varphi((y \wedge z) \to (x \wedge z))$. Notice that $x \to y \leq (x \wedge z) \to (y \wedge z)$ and $y \to x \leq (y \wedge z) \to (x \wedge z)$, we get $\varphi(x \to y) \geq \varphi((x \wedge z) \to (y \wedge z))$ and $\varphi(y \to x) \geq \varphi((y \wedge z) \to (x \wedge z))$, and so $d_\varphi(x, y) = \varphi(x \to y) \oplus \varphi(y \to x) \geq \varphi((x \wedge z) \to (y \wedge z)) \oplus \varphi((y \wedge z) \to (x \wedge z)) = d_\varphi(x \wedge z, y \wedge z)$.

(4) Using Proposition 3.3 (7), we have $\varphi((x \odot z) \to (y \odot z)) \leq \varphi(x \to y)$ and $\varphi((y \odot z) \to (x \odot z)) \leq \varphi(y \to x)$ for any $x, y, z \in L$. Therefore $d_\varphi(x, y) = \varphi(x \to y) \oplus \varphi(y \to x) \geq \varphi((x \odot z) \to (y \odot z)) \oplus \varphi((y \odot z) \to (x \odot z)) = d_\varphi(x \odot z, y \odot z)$.

(5) Since $x \to y \leq x \to (y \vee z) = (x \vee z) \to (y \vee z)$ and $y \to x \leq (y \vee z) \to (x \vee z)$ for any $x, y, z \in L$, then we get that $\varphi(x \to y) \geq \varphi((x \vee z) \to (y \vee z))$ and $\varphi(y \to x) \geq \varphi((y \vee z) \to (x \vee z))$, and so $d_\varphi(x, y) = \varphi(x \to y) \oplus \varphi(y \to x) \geq \varphi((x \vee z) \to (y \vee z)) \oplus \varphi((y \vee z) \to (x \vee z)) = d_\varphi(x \vee z, y \vee z)$.

(6) For any $x, y \in L$, we have $\varphi(x) \ominus \varphi(y) \leq \varphi(y \to x)$ and $\varphi(y) \ominus \varphi(x) \leq \varphi(x \to y)$, and so $(\varphi(x) \ominus \varphi(y)) \oplus (\varphi(y) \ominus \varphi(x)) \leq \varphi(y \to x) \oplus \varphi(x \to y) = d_\varphi(x, y)$.

**Proposition 3.9.** *Let $\varphi_1 : L_1 \to M$ and $\varphi_2 : L_2 \to M$ be two pseudo MV-valuations on BL-algebras $L_1$ and $L_2$, respectively. Then $(L_1 \times L_2, d^*)$ is a pseudo MV-metric space, where*

$$d^*((x, y), (w, v)) = d_{\varphi_1}(x, w) \oplus d_{\varphi_2}(y, v),$$

*for any $(x, y), (w, v) \in L_1 \times L_2$.*

**Proof**. According to Theorem 3.7, we get that $d_{\varphi_1}$ and $d_{\varphi_1}$ are pseudo MV-metrics on $L_1$ and $L_2$, respectively. And so $d^*((x, y), (x, y)) = d_{\varphi_1}(x, x) \oplus d_{\varphi_2}(y, y) = 0$ and $d^*((x, y), (w, v)) = d_{\varphi_1}(x, w) \oplus d_{\varphi_2}(y, v) = d_{\varphi_1}(w, x) \oplus d_{\varphi_2}(v, y) = d^*_\varphi((w, v), (x, y))$. Now let $(x, y), (a, b), (w, v) \in L_1 \times L_2$, we get $d^*((x, y), (a, b)) \oplus d^*((a, b), (w, v)) = (d_{\varphi_1}(x, a) \oplus d_{\varphi_2}(y, b)) \oplus (d_{\varphi_1}(a, w) \oplus d_{\varphi_2}(b, v)) = (d_{\varphi_1}(x, a) \oplus d_{\varphi_1}(a, w)) \oplus (d_{\varphi_2}(y, b) \oplus d_{\varphi_2}(b, v)) \geq d_{\varphi_1}(x, w) \oplus d_{\varphi_2}(y, v) = d^*((x, y), (\omega, \nu))$. Hence $(L_1 \times L_2, d^*)$ is a pseudo MV-metric space.

**Proposition 3.10.** *Let $d_\varphi$ be the pseudo MV-metric induced by a pseudo MV-valuation $\varphi$. Then $(L \times L, d)$ is a pseudo MV-metric space, where*

$$d((x, y), (w, v)) = d_\varphi(x, w) \vee d_\varphi(y, v),$$

*for any $(x, y), (\omega, \nu) \in L \times L$.*

**Proof**. The proof is similar to that of Proposition 3.9.

A function between two metric spaces is called an isometry if it preserves distances. Let $\varphi_1 : L_1 \to M$ and $\varphi_2 : L_2 \to M$ be two pseudo MV-valuations on BL-algebras $L_1$ and $L_2$, respectively. Then the function $f : L_1 \to L_2$ will be called pseudo MV-valuation preserving if $\varphi_2(f(x)) = \varphi_1(x)$, for any $x \in L_1$.

**Theorem 3.11.** *Let $L_1$, $L_2$ be BL-algebras, $\varphi_1 : L_1 \to M$ and $\varphi_2 : L_2 \to M$ be pseudo MV-valuations. If $f : L_1 \to L_2$ is a homomorphism, then the following are equivalent:*

(1) *$f$ is pseudo MV-valuation preserving;*
(2) *$f$ is an isometry.*

**Proof**. Assume that $f$ is pseudo MV-valuation preserving, then for any $x, y \in L_2$, we have $d_{\varphi_2}(f(x), f(y)) = \varphi_2(f(x) \to f(y)) \oplus \varphi_2(f(y) \to f(x)) = \varphi_2(f(x \to y)) \oplus \varphi_2(f(y \to x)) = \varphi_1(x \to y) \oplus \varphi_1(y \to x) = d_{\varphi_1}(x, y)$, hence $f$ is an isometry.

Conversely, suppose that $f$ is an isometry, then for any $x \in L_1$, we get that $\varphi_1(x) = d_{\varphi_1}(x, 1) = d_{\varphi_2}(f(x), f(1)) = \varphi_2(f(x))$. Therefore $f$ is pseudo MV-valuation preserving.

**Proposition 3.12.** *Let $L_1$, $L_2$ be BL-algebras, $\varphi : L_1 \to M$ a pseudo MV-valuation on $L_1$ and $f : L_1 \to L_2$ be an epimorphism. Then $f(\varphi) : L_2 \to M$ is a pseudo MV-valuation on $L_2$, where $f(\varphi)$ is defined by $f(\varphi)(y) = \inf\{\varphi(x) | f(x) = y, x \in L_1\}$ for any $y \in L_2$.*

**Proof**. Since $\varphi : L_1 \to M$ is a pseudo MV-valuation on $L_1$ and $f$ is an epimorphism from $L_1$ to $L_2$, it follows that $f(\varphi)(1_{L_2}) = \inf\{\varphi(x) | f(x) = 1_{L_2}, x \in L_1\} = \varphi(1_{L_1}) = 0$. For any $a, b \in L_2$, $f(\varphi)(a) \oplus f(\varphi)(a \to b) = \inf\{\varphi(x) | f(x) = a, x \in L_1\} \oplus \inf\{\varphi(x \to y) | f(x \to y) = a \to b, x, y \in L_1\} \geq \inf\{\varphi(x) \oplus \varphi(x \to y) | f(x) = a, f(y) = b, x, y \in L_1\} \geq \inf\{\varphi(y) | f(y) = b, y \in L\} = f(\varphi)(b)$, thus $f(\varphi)$ is a pseudo MV-valuation on $L_2$.

**Proposition 3.13.** *Let $L_1$, $L_2$ be BL-algebras, $\varphi : L_2 \to M$ a pseudo MV-valuation on $L_2$ and $f : L_1 \to L_2$ a homomorphism. Then $f^{-1}(\varphi) : L_1 \to M$ is a pseudo MV-valuation on $L_1$, where $f(\varphi)$ is defined by $f^{-1}(\varphi)(x) = \varphi(f(x))$ for any $x \in L_1$.*

**Proof**. Since $f : L_1 \to L_2$ is a homomorphism, then $f^{-1}(\varphi)(1_{L_1}) = \varphi(f(1_{L_1})) = \varphi(1_{L_2}) = 0$. For any $x, y \in L_1$, we get that $f^{-1}(\varphi)(x \to_{L_1} y) = \varphi(f(x \to_{L_1} y)) = \varphi(f(x) \to_{L_2} f(y)) \geq \varphi(f(y)) \ominus \varphi(f(x))$, thus $\varphi(f(x)) \leq \varphi(f(x)) \oplus f^{-1}(\varphi)(x \to_{L_1} y)$, and so $f^{-1}(\varphi)$ is a pseudo MV-valuation on $L_1$.

**Acknowledgements**

The works described in this paper are partially supported by the Higher Education Key Scientific Research Program Funded by Henan Province (No. 18A110008, 18A630001, 18A110010), Undergraduate Innovation Foundation Project of Anyang Normal University (No. ASCX/2017-Z103) and Provincial Undergraduate Innovation and Entrepreneurship Training Program of Henan Province (201710479023).

## References


[1] Hájek P. Metamathematics of fuzzy logic. Kluwer, Dordrecht, 1998.

[2] Turunen E. RS-BL-algebras are MV-algebras. Iranian Journal of Fuzzy Systems, 2016,13(6): 153–154.

[3] Lele C, Nganou J B. MV-algebras derived from ideals in BL-algebras. Fuzzy Sets & Systems, 2013, 218(4): 103–113.

[4] Rachunek J, Salounova D. Ideals and involutive filters in generalizations of fuzzy structures. Fuzzy Sets & Systems, 2017, 218(4): 103–113.

[5] Yang Y, Xin X. On characterizations of BL-algebras via implicative ideals. Italian Journal of Pure and Applied Mathematics, 2017, 37: 493–506.

[6] Buşneag D. Hilbert algebras with valuations. Discrete Mathematics, 2003, 263: 11–24.

[7] Buşneag D. Valuations on residuated lattices. Annals of the University of Craiova. Mathematics and Computer Science Series, 2007, 34: 21–28.

[8] Jun Y B, Ahn S S, Roh E H. BCC-algebras with pseudo-valuations. Applied Mathematical Sciences, 2012, 26(2): 243–252.

[9] Zhan J, Jun Y B. (Implicative) pseudo-valuations on $R_0$-algebras. U.P.B. Sci. Bull., Series A. 2013, 75(4): 101–112.

[10] Yang Y, Xin X. EQ-Algebras with pseudo pre-valuations. Italian Journal of Pure & Applied Mathematics, 2017(37):29-48.

[11] Yang Y, Lu L, Wang Q. Pseudo valuations on hoop-algebras with respect to filters. EPH-International Journal of Mathematics and Statistics, 2017, 11(2): 12–19.

[12] Cignoli R L, D'Ottaviano I M, Mundici D. Algebraic foundations of many-valued reasoning, Springer Science & Business Media, 2013.