Optimal Convex Combination Bounds of Arithmetic and Second Seiffert Means for Neuman-Sándor Mean

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Abstract. In this paper, we present the least value α and the greatest value β such that the double inequality

$$\alpha A(a,b)+(1-\alpha)T(a,b)<M(a,b)<\beta A(a,b)+(1-\beta)T(a,b)$$

hold for all $a,b>0$ with $a\neq b$, where $A(a,b),M(a,b)$ and $T(a,b)$ are the arithmetic, Neuman-Sándor and second Seiffert means of a and b, respectively.

1. Introduction

For $a,b>0$ with $a\neq b$ the Neuman-Sándor mean $M(a,b)$[1] was defined by

$$M(a,b)=\frac{a-b}{2\sinh^{-1}(\frac{a-b}{a+b})}, \tag{1.1}$$

where $\sinh^{-1}(x)=\log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the Neuman-Sándor mean has been the subject of intensive research. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a,b)$ can be found in the literature [1,2].

Let $H(a,b)=(2ab)/(a+b),\ G(a,b)=\sqrt{ab},\ L(a,b)=(a-b)/(\log a-\log b),\ P(a,b)=(a-b)/(4\tan^{-1}\sqrt{a/b}-\pi),\ A(a,b)=(a+b)/2,\ T(a,b)=(a-b)/[2\tan^{-1}(a-b)/(a+b)],\ Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $C(a,b)=(a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic and contra-harmonic means of a and b, respectively. Then

$$\begin{aligned}&\min\{a,b\}<H(a,b)<G(a,b)<L(a,b)<P(a,b)<A(a,b)\\&<M(a,b)<T(a,b)<Q(a,b)<C(a,b)<\max\{a,b\}\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$.

Neuman and Sándor [1, 2] proved that the inequalities

$$\frac{\pi}{4\log(1+\sqrt{2})}I(a,b)<M(a,b)<\frac{A(a,b)}{\log(1+\sqrt{2})},$$

$$\sqrt{2T^2(a,b)-Q^2(a,b)}<M(a,b)<\frac{T^2(a,b)}{Q^2(a,b)},$$

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$$\begin{gathered} H(T(a,b),A(a,b)) < M(a,b) < L(A(a,b),Q(a,b)), \\ T(a,b) > H(M(a,b),Q(a,b)),\ M(a,b) < \frac{A^2(a,b)}{P(a,b)}, \\ A^{2/3}(a,b)Q^{1/3}(a,b) < M(a,b) < \frac{2A(a,b)+Q(a,b)}{3}, \\ \sqrt{A(a,b)T(a,b)} < M(a,b) < \sqrt{A^2(a,b)+T^2(a,b)}, \\ \frac{G(x,y)}{G(1-x,1-y)} < \frac{L(x,y)}{L(1-x,1-y)} < \frac{P(x,y)}{P(1-x,1-y)} \\ < \frac{A(x,y)}{A(1-x,1-y)} < \frac{M(x,y)}{M(1-x,1-y)} < \frac{T(x,y)}{T(1-x,1-y)}, \\ \frac{1}{A(1-x,1-y)} - \frac{1}{A(x,y)} < \frac{1}{M(1-x,1-y)} - \frac{1}{M(x,y)} < \frac{1}{T(1-x,1-y)} - \frac{1}{T(x,y)}, \\ A(x,y)A(1-x,1-y) < M(x,y)M(1-x,1-y) < T(x,y)T(1-x,1-y) \end{gathered}$$

hold for all $a,b>0$ and $x,y \in (0,1/2)$ with $a \neq b$ and $x \neq y$.

Li et al. [3] showed that the double inequality

$$L_{p_0}(a,b) < M(a,b) < L_2(a,b)$$

holds for all $a,b>0$ with $a \neq b$, where $L_p(a,b) = [(a^{p+1}-b^{p+1})/((p+1)(a-b))]^{1/p} (p \neq -1, 0)$, $L_0(a,b) = 1/e(a^a/b^b)^{1/(a-b)}$ and $L_{-1}(a,b) = (a-b)/(\log a - \log b)$ is the p-th generalized logarithmic mean of a and b, and $p_0 = 1.843\cdots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

In [4], Neuman proved that the double inequalities

$$\alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b)$$

and

$$\lambda C(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu C(a,b) + (1-\mu)A(a,b)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha \leq [1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots, \beta \geq 1/3, \lambda \leq [1-\log(1+\sqrt{2})]/\log(1+\sqrt{2}) = 0.1345\cdots$ and $\mu \geq 1/6$.

In [5], Yuming Chu etc proved that the double inequalities

$$\alpha_1 L(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 L(a,b) + (1-\beta_1)Q(a,b)$$

and

$$\alpha_2 L(a,b) + (1-\alpha_2)C(a,b) < M(a,b) < \beta_2 L(a,b) + (1-\beta_2)C(a,b)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha_1 \geq 2/5, \beta_1 \leq 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\cdots, \alpha_2 \geq 5/8$ and $\beta_2 \leq 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\cdots$.

In addition, inequalities for quotients involving the Neuman-Sándor mean M(a,b) were obtained in [6].

The main purpose of this paper is to find the least value α and the greatest value β such that the double inequality

$$\alpha A(a,b) + (1-\alpha)T(a,b) < M(a,b) < \beta A(a,b) + (1-\beta)T(a,b)$$

holds for all $a,b>0$ with $a \neq b$. All numerical computations are carried out using the mathematical calculation software.

2. Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

Lemmas 1. Let $\mu = 1/(4-\pi)[4-\pi/\log(1+\sqrt{2})] = 0.5074\cdots, p \in \{1/2, \mu\}$, and $\omega_p(t) = (p-1)^3t^4 + (1-p)^2(1-10p)t^3 + (1-p)(8p^2-14p+1)t^2 + (4p^2-2p+3)t + 2(2p+1)$. Then $\omega_p(t) > 0$ holds for all $t \in (0,1)$.

Proof. Simple computations lead to

$$\lim_{t\to 0^+} \omega_p(t) = 2(2p+1) > 0, \ \lim_{t\to 1^-} \omega_p(t) = (2-p)(19p^2-12p+4) > 0, \tag{2.1}$$

$$\lim_{t\to 0^+} \omega_p'(t) = 4p^2-2p+3 > 0, \ \lim_{t\to 1^-} \omega_p'(t) = (3-2p)(25p^2-24p+4) < 0, \tag{2.2}$$

$$\lim_{t\to 0^+} \omega_p''(t) = 2(1-p)(8p^2-14p+1) < 0, \tag{2.3}$$

and

$$\omega_p'''(t) = 6[4(p-1)^3t + (1-p)^2(1-10p)] < 0 \tag{2.4}$$

for $t \in (0,1)$. (2.3) and (2.4) imply that $\omega_p'(t)$ is strictly decreasing in $(0,1)$. It follows from (2.2) and the monotonicity of $\omega_p'(t)$ that there exists $t_0 \in (0,1)$ such that $\omega_p'(t) > 0$ for $t \in (0,t_0)$ and $\omega_p'(t) < 0$ for $t \in (t_0,1)$, hence $\omega_p(t)$ is strictly increasing in $(0,t_0)$ and strictly decreasing in $(t_0,1)$. Therefore the conclusion of lemma 1 is educed from (2.1) the monotonicity of $\omega_p(t)$. □

Lemmas 2. Let $\mu = 1/(4-\pi)[4-\pi/\log(1+\sqrt{2})] = 0.5074\cdots, p \in \{1/2, \mu\}$, and $\upsilon_p(t) = 2[2(1-p)^2t^3 + 5(1-p)^2t^2 + 2(p^2-3p+1)t - (2p+1)]$. Then $\upsilon_p(t) < 0$ holds for all $t \in (0,1)$.

Proof. Simple computations yield

$$\lim_{t\to 0^+} \upsilon_p(t) = -2(2p+1) < 0, \ \lim_{t\to 1^-} \upsilon_p(t) = 2(p-2)(9p-4) < 0, \tag{2.5}$$

$$\lim_{t\to 0^+} \upsilon_p'(t) = 4(p^2-3p+1) < 0, \ \lim_{t\to 1^-} \upsilon_p'(t) = 4(9p^2-19p+9) > 0, \tag{2.6}$$

and

$$\upsilon_p''(t) = 4(1-p)^2(6t+5) > 0 \tag{2.7}$$

holds for all $t \in (0,1)$. From (2.7) we know that $\upsilon_p'(t)$ is strictly increasing in $(0,1)$.

It follows from (2.6) and the monotonicity of $\upsilon_p'(t)$ that there exists $t_1 \in (0,1)$ such that $\upsilon_p'(t) < 0$ for $t \in (0,t_1)$ and $\upsilon_p'(t) > 0$ for $t \in (t_1,1)$, hence $\upsilon_p(t)$ is strictly decreasing in $(0,t_1)$ and strictly increasing in $(t_1,1)$. Therefore the conclusion of lemma 2 is elicited from (2.5) and the monotonicity of $\upsilon_p(t)$. □

Lemmas 3. Let $\mu = 1/(4-\pi)[4-\pi/\log(1+\sqrt{2})] = 0.5074\cdots$, and $L_\mu(t) = (1-\mu)^6t^7 + 2(1-\mu)^4(10\mu^2-11\mu-7)t^6 + (1-\mu)^4(116\mu^2-48\mu-93)t^5 + 4(1-\mu)^2(40\mu^4-116\mu^3+36\mu^2+99\mu-51)t^4 + (1-\mu)^2(64\mu^4-304\mu^3+40\mu^2+480\mu-185)t^3 - 2(32\mu^5-16\mu^4-240\mu^3+398\mu^2-181\mu+15)t^2 - (64\mu^4-336\mu^3+380\mu^2-16\mu-53)t + 8(1+2\mu)(1-2\mu)(3-2\mu)$. Then there exists $\eta_2 \in (0,1)$ such that $L_\mu(t) < 0$ for $t \in (0,\eta_2)$ and $L_\mu(t) > 0$ for $t \in (\eta_2,1)$.

Proof. By calculating first-sixth derived functions of $L_\mu(t)$ and the numerical computations we know that $L_\mu^{(6)}(t) < 0$ for $t \in (0,1)$, and $L_\mu(0) < 0$, $L_\mu(1) > 0$, $L_\mu'(0) > 0$, $L_\mu'(1) > 0$, $L_\mu''(0) > 0$, $L_\mu''(1) < 0$, $L_\mu'''(0) > 0$, $L_\mu'''(1) < 0$, $L_\mu^{(4)}(0) < 0$, $L_\mu^{(5)}(0) < 0$. Apparently $L_\mu^{(4)}(0) < 0$, $L_\mu^{(5)}(0) < 0$ and $L_\mu^{(6)}(t) < 0$ imply that $L_\mu'''(t)$ is strictly decreasing in $(0,1)$.

It follows from $L_\mu'''(0) > 0$ and $L_\mu'''(1) < 0$ together with the monotonicity of $L_\mu'''(t)$ that there exists $\eta_0 \in (0,1)$ such that $L_\mu'''(t) > 0$ for $t \in (0,\eta_0)$ and $L_\mu'''(t) < 0$ for $t \in (\eta_0,1)$, so

$L''_\mu(t)$ is strictly increasing in $(0, \eta_0)$ and strictly decreasing in $(\eta_0, 1)$. From $L''_\mu(0) > 0$ and $L''_\mu(1) < 0$ together with the monotonicity of $L''_\mu(t)$ we know that there exists $\eta_1 \in (\eta_0, 1)$ such that $L''_\mu(t) > 0$ for $t \in (0, \eta_1)$ and $L''_\mu(t) < 0$ for $t \in (\eta_1, 1)$, hence $L'_\mu(t)$ is strictly increasing in $(0, \eta_1)$ and strictly decreasing in $(\eta_1, 1)$. $L'_\mu(0) > 0$ and $L'_\mu(1) > 0$ together with the monotonicity of $L'_\mu(t)$ imply that $L'_\mu(t) > 0$ for $t \in (0, 1)$, thus $L_\mu(t)$ is strictly increasing in $(0, 1)$. Therefore the conclusion of lemma 3 follows from $L_\mu(0) < 0$ and $L_\mu(1) > 0$ together with the monotonicity of $L_\mu(t)$.

□

3. Main Results

theorem. The double inequality

$$\alpha A(a,b) + (1-\alpha)T(a,b) < M(a,b) < \beta A(a,b) + (1-\beta)T(a,b) \tag{3.1}$$

holds true for $a, b > 0$ with $a \neq b$ if and only if $\alpha \geq 1/(4-\pi)[4 - \pi/\log(1+\sqrt{2})] = 0.5074\cdots$ and $\beta \leq 1/2$.

Proof. Let $\mu = 1/(4-\pi)[4 - \pi/\log(1+\sqrt{2})] = 0.5074\cdots$. Firstly we prove that

$$\frac{1}{2}[A(a,b) + T(a,b)] > M(a,b), \tag{3.2}$$

and

$$\mu A(a,b) + (1-\mu)T(a,b) < M(a,b). \tag{3.3}$$

Without loss of generality, we assume that $a > b > 0$. Let $x = (a-b)/(a+b) \in (0,1)$ and $p \in \{1/2, \mu\}$. Then

$$\frac{M(a,b)}{A(a,b)} = \frac{x}{\sinh^{-1}(x)}, \quad \frac{T(a,b)}{A(a,b)} = \frac{x}{\tan^{-1} x}, \tag{3.4}$$

and

$$\frac{pA(a,b) + (1-p)T(a,b) - M(a,b)}{A(a,b)} = \frac{E_p(x)}{\log(x+\sqrt{1+x^2})\tan^{-1} x}, \tag{3.5}$$

where

$$E_p(x) = p\tan^{-1} x \log(x+\sqrt{1+x^2}) + (1-p)x\log(x+\sqrt{1+x^2}) - x\tan^{-1} x. \tag{3.6}$$

Some tedious, but not difficult, calculations lead to

$$\lim_{x\to 0^+} E_p(x) = 0, \tag{3.7}$$

$$\lim_{x\to 1^-} E_p(x) = [(\frac{\pi}{4} - 1)p + 1]\log(1+\sqrt{2}) - \frac{\pi}{4}, \tag{3.8}$$

$$E'_p(x) = \frac{[1+(1-p)x^2]G_p(x)}{1+x^2}, \tag{3.9}$$

where

$$G_p(x) = \frac{p(\tan^{-1} x - px + x)\sqrt{1+x^2} - (1+x^2)\tan^{-1} x - x}{1+(1-p)x^2} + \log(x+\sqrt{1+x^2}), \tag{3.10}$$

$$\lim_{x\to 0^+} G_p(x) = 0, \tag{3.11}$$

$$\lim_{x\to 1^-} G_p(x) = \log(1+\sqrt{2}) + \frac{(\pi-4)\sqrt{2}p + 2(2\sqrt{2}-\pi-2)}{4(2-p)}, \tag{3.12}$$

$$G'_p(x) = \frac{px[(1-2p) + (1-p)x^2 + 2\sqrt{1+x^2}]H_p(x)}{[1+(1-p)x^2]^2\sqrt{1+x^2}}, \tag{3.13}$$

where

$$H_p(x) = \frac{(1-p)^2x^4 + (3-2p^2-p)x^2 - 2\sqrt{1+x^2}+2}{px[(1-2p)+(1-p)x^2+2\sqrt{1+x^2}]} - \tan^{-1} x, \tag{3.14}$$

$$\lim_{x\to 0^+} H_p(x) = 0, \tag{3.15}$$

$$\lim_{x\to 1^-} H_p(x) = \frac{2(3-\sqrt{2})-p(p+3)}{p(2+2\sqrt{2}-3p)} - \frac{\pi}{4}, \tag{3.16}$$

and

$$H_p'(x) = \frac{K_p(x)}{px^2(1+x^2)[(1-2p)+(1-p)x^2+2\sqrt{1+x^2}]^2}, \tag{3.17}$$

where

$$\begin{aligned} K_p(x) = \ & (p-1)^3x^8 + (1-p)^2(1-10p)x^6 + (1-p)(8p^2-14p+1)x^4 \\ & +(4p^2-2p+3)x^2 + 2(2p+1) + [4(p-1)^2x^6 + 10(1-p)^2 \\ & \cdot(1-10p)x^4 + 4(p^2-3p+1)x^2 - 2(2p+1)]\sqrt{1+x^2}. \end{aligned} \tag{3.18}$$

Let $x = \sqrt{t}$ $(t \in (0,1))$, then

$$K_p(x) = \omega_p(t) + \upsilon_p(t)\sqrt{1+t} = \frac{tL_p(t)}{\omega_p(t) - \upsilon_p(t)\sqrt{1+t}}, \tag{3.19}$$

where $\omega_p(t)$ and $\upsilon_p(t)$ are defined as in lemmas 1 and 2, respectively, and

$$\begin{aligned} L_p(t) = \ & (1-p)^6t^7 + 2(1-p)^4(10p^2-11p-7)t^6 + (1-p)^4(116p^2-48p-93)t^5 \\ & +4(1-p)^2(40p^4-116p^3+36p^2+99p-51)t^4 + (1-p)^2(64p^4-304p^3 \\ & +40p^2+480p-185)t^3 - 2(32p^5-16p^4-240p^3+398p^2-181p+15)t^2 \\ & -(64p^4-336p^3+380p^2-16p-53)t + 8(1+2p)(1-2p)(3-2p). \end{aligned} \tag{3.20}$$

Now we distinguish between two cases:

Case 1. $p = 1/2$. (3.20) leads to

$$L_{1/2}(t) = \frac{1}{64}t(t+2)^2[t^4 + 84t^2(1-t) + 104t(1-t) + 8(3t+8)] > 0, \tag{3.21}$$

holds for all $t \in (0,1)$. This fact and (3.19), (3.17) together with lemmas 1 and 2 imply that $H_{1/2}'(x) > 0$ for $x \in (0,1)$, hence $H_{1/2}(x)$ is strictly increasing in $(0,1)$. Therefore the inequality (3.2) follows from (3.5), (3.7), (3.9), (3.11), (3.13) and (3.15) together with the monotonicity of $H_{1/2}(x)$.

Case 2. $p = \mu$. Here (3.20) becomes $L_\mu(t)$, which is defined as in lemma 3. By (3.19) and the conclusions of lemmas $1-3$ we confirm that $K_\mu(x) < 0$ for $x \in (0, x_0)$ and $K_\mu(x) > 0$ for $x \in (x_0, 1)$, where $x_0 = \sqrt{\eta_2}$. This fact and (3.18) imply that $H_\mu'(x) < 0$ for $x \in (0, x_0)$ and $H_\mu'(x) > 0$ for $x \in (x_0, 1)$, hence $H_\mu(x)$ is strictly decreasing in $(0, x_0)$ and strictly increasing in $(x_0, 1)$.

Notice that (3.8), (3.12) and (3.16) become

$$\lim_{x\to 1^-} E_\mu(x) = 0, \ \lim_{x\to 1^-} G_\mu(x) = 0.0033\cdots > 0, \ \lim_{x\to 1^-} H_\mu(x) = 0.0442\cdots > 0, \tag{3.22}$$

respectively. It follows from (3.22), (3.15), (3.13), (3.11), (3.9) and (3.7) together with the monotonicity of $H_\mu(x)$ that

$$E_\mu(x) < 0 \tag{3.23}$$

for $x \in (0,1)$. Therefore the inequality (3.3) follows from (3.5) and (3.23).

Finally, we prove that $\mu A(a,b) + (1-\mu)T(a,b)$ is the best possible lower convex combination bound and $1/2[A(a,b) + T(a,b)]$ is the best possible upper convex combination bound of the arithmetic and the second Seiffert means for the Neuman-Sándor mean.

Equations (3,4) lead to

$$\frac{T(a,b)-M(a,b)}{T(a,b)-A(a,b)} = \frac{x/\tan^{-1}x - x/\sinh^{-1}(x)}{x/\tan^{-1}x-1} = R(x). \tag{3.24}$$

From (3.23) one has

$$\lim_{x\to 1^-} R(x) = \mu, \tag{3.25}$$

and

$$\lim_{x\to 0^+} R(x) = \frac{1}{2}. \tag{3.26}$$

If $\alpha < \mu$, then (3.24) and (3.25) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $M(a,b) < \alpha A(a,b) + (1-\alpha)T(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (\delta_1, 1)$.

If $\beta > 1/2$, then (3.24) and (3.26) lead to the conclusion that there exists $0 < \delta_2 < 1$ such that $M(a,b) > \beta A(a,b) + (1-\beta)T(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (0, \delta_2)$.

□

Remark. In [7], we proved that the double inequality

$$\alpha G(a,b) + (1-\alpha)T(a,b) < M(a,b) < \beta G(a,b) + (1-\beta)T(a,b) \tag{3.27}$$

holds true for $a,b > 0$ with $a \neq b$ if and only if $\alpha \geq 1/5$ and $\beta \leq 1 - \pi/[4\log(1+\sqrt{2})] = 0.108893\cdots$.

The bounds in the double inequalities (3.1) and (3.27) are not comparable to each other. In fact, if we let $a > b > 0$ and $x = \sqrt{a/b} > 1$, and notate $\lambda = 1 - \pi/[4\log(1+\sqrt{2})]$ and $\omega = 1/(4-\pi)[4 - \pi/\log(1+\sqrt{2})]$, then

$$[\frac{1}{2}A(a,b) + \frac{1}{2}T(a,b)] - [\lambda G(a,b) + (1-\lambda)T(a,b)] = \frac{b}{\tan^{-1}\dfrac{x^2-1}{x^2+1}} F_1(x) \tag{3.28}$$

and

$$[\omega A(a,b) + (1-\omega)T(a,b)] - [\frac{1}{5}G(a,b) + \frac{4}{5}T(a,b)] = \frac{b}{\tan^{-1}\dfrac{x^2-1}{x^2+1}} F_2(x), \tag{3.29}$$

where

$$F_1(x) = \left(\frac{x^2}{4} - \lambda x + \frac{1}{4}\right)\tan^{-1}\frac{x^2-1}{x^2+1} + \frac{2\lambda-1}{4}(x^2-1) \tag{3.30}$$

and

$$F_2(x) = \left[\frac{\omega(x^2+1)}{2} - \frac{x}{5}\right]\tan^{-1}\frac{x^2-1}{x^2+1} - \frac{\omega(x^2-1)}{2} + \frac{x^2-1}{10}, \tag{3.31}$$

respectively. Simple computations yield

$$F_1(1) = F_1'(1) = F_1''(1) = 0,\ F_1'''(1) = 5\lambda - 1 = -0.4555\cdots < 0, \tag{3.32}$$

$$\lim_{x\to+\infty} F_1(x) = \lim_{t\to 0^+} \frac{(t^2 - 4\lambda t + 1)\tan^{-1}\dfrac{1-t^2}{t^2+1} + (1-2\lambda)(t^2-1)}{4t^2} = +\infty, \tag{3.33}$$

$$F_2(1) = F_2'(1) = F_2''(1) = 0,\ F_2'''(1) = 1 - 2\omega = -0.0148\cdots < 0, \tag{3.34}$$

and

$$\lim_{x\to+\infty} F_2(x) = \lim_{t\to 0^+} \frac{[5\omega(t^2+1) - 2t]\tan^{-1}\dfrac{1-t^2}{t^2+1} + (1-5\omega)(1-t^2)}{10t^2} = +\infty. \tag{3.35}$$

Equations (3.28), (3.32) and (3.33) imply that there exist small enough $\delta_1 > 0$ and large enough $X_1 > 0$ such that $1/2A(a,b)+1/2T(a,b) < \lambda G(a,b)+(1-\lambda)T(a,b)$ for $\sqrt{a/b} \in (1, 1+\delta_1)$, and $1/2A(a,b) + 1/2T(a,b) > \lambda G(a,b) + (1-\lambda)T(a,b)$ for $\sqrt{a/b} \in (X_1, +\infty)$.

Equations (3.29), (3.34) and (3.35) imply that there exist small enough $\delta_2 > 0$ and large enough $X_2 > 0$ such that $\omega A(a,b) + (1-\omega)T(a,b) < 1/5G(a,b) + 4/5T(a,b)$ for $\sqrt{a/b} \in (1, 1+\delta_2)$, and $\omega A(a,b) + (1-\omega)T(a,b) > 1/5G(a,b) + 4/5T(a,b)$ for $\sqrt{a/b} \in (X_2, +\infty)$.

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