Some Characterizations of Similar curves in 4D-Galilean space

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Abstract— *In this work, we introduce families of similar curves in 4D-Galilean space with variable transformation. Also, we obtain some characterizations of this family and some theorems. Moreover, Using new Serret-Frenet vectors of these curves, we obtain some new relationships between with non-zero curvatures of the similar partner curves and unit tangent vector T of ξ satisfies a vector differential equation of fourth order in 4D-Galilean space.*
Keywords— *Galilean space, Frenet frame, Similar curve,differential equation.*
Mathematics Subject Classification: *14H45, 53A04.*

I. INTRODUCTION

In differential geometry, special curves have an important role. Especially the partner curves, i.e., the curves which are related to each other at the corresponding points, have attracted the attention of many mathematicians. Well-known partner curves are the Bertrand curves, which are defined by the property that at the corresponding points of two space curves the principal normal vectors are common. Bertrand partner curves are studied in [1, 2, 3, 10]. Ravani and Ku transported the notion of Bertrand curves to the ruled surfaces and called them Bertrand offsets.

In recent years, researchers have begun to investigate curves and surfaces in the Galilean spaces and thereafter pseudo- Galilean space. The theory of the curves in Galilean spaces is extensively studied in Röschel (1986). In this space we refer; about spherical curves in G3, Ergüt and Öğrenmiş (2009), Ogrenmiş et al. (2007); on Bertrand curves Öğrenmiş et al. (2009). It is safe to report that a good amount of researches have also been done in pseudo-Galilean space by the aid of the interesting paper by Divjak (1998); and thereafter classical differential geometry papers Divjak and Milin-Sipus (2003 and 2008) and Öğrenmiş and Ergüt (2009) [4, 5, 6, 7].

Recently, Similar curves have been introduced a new type of special curves in E^3 for which the arc-length parameters have a relationship and between the space curves α and α_* such that, at the corresponding points of the curves, the tangent lines of α coincides with the tagent of α_*, then α is a called a similar curve, and α_* similar partner curve of α [8, 9].

In this work, in the light of the existing literature we extend aspects of classical differential geometry topics to 4D-Galilean space, we obtain a family of curves and call them a family of similar curves in 4D-Galilean space with variable transformation . Obtain some characterizations of these families and some theorems. And then we express that the families of curves with vanishing curvatures forms we obtain a family of similar curves in 4D-Galilean space.

II. PRELIMINARIES

Let 4D-Galilean space be the 4-dimensional Galilean space and ξ is a curve in 4D-Galilean space given cordinate form $\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t))$. Where $\xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)$ are smoot functions. For any vectors $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$ in 4D-Galilean space, The scalar inner product of two vectors are defined by

$$\langle u . v \rangle = \begin{cases} u_1 v_1 & ; \; u_1 \neq 0 \;\; or \;\; v_1 \neq 0 \\ u_2 v_2 + u_3 v_3 + u_4 v_4 & ; \; u_1 = 0 \;\; or \;\; v_1 = 0 \end{cases} \quad (1)$$

Then,we define the Galilean cross product in 4D-Galilean space for vectors $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2, v_3, v_4)$ and $w = (w_1, w_2, w_3, w_4)$ as fallows:

$$u \Lambda v \Lambda w = \begin{bmatrix} 0 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}$$

where $e_i \quad (1 \leq i \leq 4)$, are the standart basis vectors. Let ξ be a curve in 4D-Galilean space, parameterized by arc-length s, given coordinate form $\xi(s) = (s, \xi_2(s), \xi_3(s), \xi_4(s))$. The tangent vector of ξ is given by $T = \xi'(s) = (1, \xi_2'(s), \xi_3'(s), \xi_4'(s))$. Since T is a unit vector, we can write $\langle T, T \rangle = 1$. Differentating above equation with respect to s, we obtain $\langle T, T' \rangle = 0$. From this, we obtain the curvature κ as follows:

$$\kappa(s) = \|T'(s)\| = \sqrt{(\xi_2'')^2 + (\xi_3'')^2 + (\xi_4'')^2}.$$

If $\kappa(s) \neq 0$, for all s∈I. Similar to space G_3, we define the principal vector

$$N(s) = \frac{T'(s)}{\kappa(s)} = \frac{(0, \xi^{2''}, \xi^{3''}, \xi^{4''})}{\kappa(s)}.$$

By the differentiation of the principal normal vector given in above equation, we have second curvature function as follows:

$$\tau(s) = \|N'(s)\|.$$

This real valued function is called torsion of the curve ξ principal binormal vector field of the ξ given by

$$B(s) = \frac{1}{\tau(s)} [0, \frac{\xi^{2''}(s)}{\kappa(s)}, \frac{\xi^{3''}(s)}{\kappa(s)}, \frac{\xi^{4''}(s)}{\kappa(s)}]$$

Principal binormal vector is orthogonal to T and N. The secondary binormal vector is defined by

$$E(s) = \eta T(s) \Lambda N(s) \Lambda B(s).$$

Where η is taken ±1 to make +1 determinant of the matrix $\{T, N, B, E\}$. The third curvature of the curve ξ is defined by

$$\sigma(s) = \langle B'(s), E(s) \rangle.$$

Then the Frenet formulae of curve ξ are given by

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix} \quad (2)$$

III. SIMILAR PARTNER CURVES IN 4D-GALILEAN SPACE

In this section, we give the definition and some characterizations of Similar curves in 4D-Galilean space with variable transformations. Before givin the characterizations, first we give the following definition and theorem.

Definition 1: Let 4D-Galilean space be the 4- dimensional Galilean space with the inner product $\langle\ .\rangle G_4$ and $\xi_{\alpha}(s_\alpha)$ and $\xi_{\beta}(s_\beta)$ be curves in 4D-Galilean space parameterized by arclengths s_α and s_β with non-zero curvatures $\{\kappa_\alpha, \tau_\alpha, \sigma_\alpha\}$, $\{\kappa_\beta, \tau_\beta, \sigma_\beta\}$ and Frenet frames $\{T_\alpha, N_\alpha, B_\alpha, E_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta, E_\beta\}$ respectively. $\xi_{\alpha}(s_\alpha)$ and $\xi_{\beta}(s_\beta)$ are called similar curve in 4D-Galilean space with variable transformation λ_β^α . And at the corresponding points of the curves, the tangent lines of $\xi_{\alpha}(s_\alpha)$ coincidet with the tangents lines of $\xi_{\beta}(s_\beta)$,

$$T_\alpha(s_\alpha) = T_\beta(s_\beta) \qquad (3)$$

and $s_\alpha = \int \lambda_\beta^\alpha ds_\beta$ of arclengths s_α and s_β, where λ_β^α is arbitrary function of arclength. It is worth nothing that $\lambda_\beta^\alpha \lambda_\alpha^\beta = 1$. If we integrate the equality (3) we obtain the following theorem.

Theorem 1: The position vectors of the family similar curves in 4D-Galilean space with variable transformation can be written in the following form,

$$\xi_{\alpha}(s_\alpha) \quad = \int T_\alpha\left(s_\alpha(s_\beta)\right) ds_\beta = \int T_\beta(s_\alpha)\lambda_\alpha^\beta ds_\beta,$$

Theorem 2: Let $\xi = \xi(s)$ be a curve parameterized by arclength s. Provided that $\xi = \xi(\varphi)$ be another parametrization of the curve with parameter $\varphi = \int \kappa(s)ds$. Then in 4D-Galilean space the unit tangent vector T of ξ satisfies a vector differential equation of fourth order as follows:

$$\left\{\left(\frac{1}{g}\right)\left[\left(\frac{1}{g}\right)\left(\frac{T''}{f}\right)' + (f\, T')\right]' + \frac{T''}{f} = 0\right\} \qquad (4)$$

where

$$f(\varphi) = \left(\frac{\tau(\varphi)}{\kappa(\varphi)}\right), g(\varphi) = \left(\frac{\sigma(\varphi)}{\kappa(\varphi)}\right), (T)' = \frac{dT}{d\varphi}, \qquad (T)'' = \frac{d^2T}{d\varphi^2}$$

Proof: If we write derivatives given in (2) according to ϕ, we obtain

$$\left(\frac{dT}{d\varphi}\right) = (\kappa N)\left(\frac{1}{\kappa}\right) = N$$

$$\left(\frac{dN}{d\phi}\right) = (-\tau B)\left(\frac{1}{\kappa}\right) = -fB$$

$$\left(\frac{dB}{d\phi}\right) = (-\tau N + \sigma E)\left(\frac{1}{\kappa}\right) = -fN + gE$$

$$\left(\frac{dE}{d\varphi}\right) = (-\sigma B)\left(\frac{1}{\kappa}\right) = -gB$$

where $f(\varphi) = \left(\frac{\tau(\varphi)}{\kappa(\varphi)}\right), g(\varphi) = \left(\frac{\sigma(\varphi)}{\kappa(\varphi)}\right)$. Then corresponding matrix form of above equatiıon can be given

$$\begin{bmatrix} T' \\ N' \\ B' \\ E' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & f & 0 & g \\ 0 & 0 & -g & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \\ E \end{bmatrix} \qquad (5)$$

Using new Frenet derivatives (5) we obtain equation (4).

Theorem 3: Let $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ be real curves in 4D-Galilean space . Then $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ are similar curves with variable transformation if and only if the principal normal vectors are the same for all curves

$$\mathrm{N}_\alpha(s_\alpha) = \mathrm{N}_\beta(s_\beta) \qquad (6)$$

under the particular variable transformation

$$\lambda_\alpha^\beta = \left(\frac{ds_\beta}{ds_\alpha}\right) = \left(\frac{\kappa_\alpha}{\kappa_\beta}\right). \qquad (7)$$

Proof: Let $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ be real similar curves with variable transformation, Then differentiating equation (3) with respect to s_β we obtain

$$\kappa_\beta(s_\beta)\mathrm{N}_\beta(s_\beta) = \kappa_\alpha(s_\alpha)\,\mathrm{N}_\alpha(s_\alpha)\frac{ds_\alpha}{ds_\beta} \qquad (8)$$

The above equation leads to equations (6) and (7).

Conversely, let $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ be real similar curves with variable transformation in 4D-Galilean space satisfaying (6) and (7). If we multiply (6) with $\kappa_\beta(s_\beta)$ and integrate the result equality with respect to (s_β), we obtain

$$\int \kappa_\beta(s_\beta)\mathrm{N}_\beta(s_\beta)ds_\beta = \int \kappa_\beta(s_\beta)\mathrm{N}_\beta(s_\beta)\frac{ds_\beta}{ds_\alpha}ds_\alpha. \qquad (9)$$

From equations (6) and (7), equation (9) take the form

$$T_\beta(s_\beta) = \int \kappa_\beta(s_\beta)\mathrm{N}_\beta(s_\beta)ds_\beta = \int \kappa_\alpha(s_\alpha)\,\mathrm{N}_\alpha(s_\alpha)ds_\alpha = T_\beta(s_\alpha)$$

which means that $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ are similar curves with variable transformation and proof is complete.

Theorem 4: Let $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ be real curves in 4D-Galilean space . Then $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ are similar curves with variable transformation if and only if the Frenet vectors B_α and B_β of the curves are the same for all curves

$$B_\alpha(s_\alpha) = B_\beta(s_\beta) \qquad (10)$$

under the particular variable transformation

$$\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{\tau_\alpha}{\tau_\beta} \qquad (11)$$

keeping equal total curvatures, i.e.,

$$\varphi_\beta(s_\beta) = \int \kappa_\beta(s_\beta) ds_\beta = \int \kappa_\alpha(s_\alpha)\, \mathrm{N}_\alpha(s_\alpha) ds_\alpha = \varphi_\alpha(s_\alpha)$$

of the arc-lengths.

Proof: Let $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ be real curves in 4D-Galilean space. Then from definition 1 and theorem 2, there exists a variable transformation of the arc-lengths. Differentiating equation (6) with respect to s_β we have

$$\sigma_\beta B_\beta = \frac{ds_\alpha}{ds_\beta} \sigma_\alpha B_\alpha$$

which gives us desired equalities (10) and

$$\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{\tau_\alpha}{\tau_\beta}$$

Conversely, let $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ be real curves in 4D-Galilean space satisfying (10) and (11). Differentiating (10) with respect to s_β we have

$$-\tau_\beta(s_\beta)\mathrm{N}_\beta(s_\beta) + \sigma_\beta(s_\beta)\mathrm{E}_\beta(s_\beta) = \frac{ds_\alpha}{ds_\beta}(\tau_\alpha(s_\alpha)\mathrm{N}_\alpha(s_\alpha) + \sigma_\alpha(s_\alpha)\mathrm{E}_\alpha(s_\alpha)) \qquad (12)$$

$$\frac{ds_\alpha}{ds_\beta} = \frac{\tau_\beta}{\tau_\alpha}, \quad \frac{ds_\beta}{ds_\alpha} = \frac{\sigma_\beta}{\sigma_\alpha} \qquad (13)$$

Then from (12) we have

$$\frac{ds_\alpha}{ds_\beta} = \frac{\tau_\beta}{\tau_\alpha} \qquad (14)$$

Using theorem 2. Then, the unit tangents $T_\beta(s_\beta) = T_\beta(s_\alpha)$ of the curves satisfy the following vector differential equation of fourth order as follows:

$$\left\{\left(\frac{1}{g_\alpha(\varphi_\alpha)}\right)\left[\left(\frac{1}{g_\alpha(\varphi_\alpha)}\right)\left(\frac{T_\alpha(\varphi_\alpha)''}{f_\alpha(\varphi_\alpha)}\right)' + \left(f_\alpha(\varphi_\alpha)T_\alpha(\varphi_\alpha)'\right)\right]' + \frac{T_\alpha(\varphi_\alpha)''}{f_\alpha(\varphi_\alpha)} = 0\right\} \quad (15)$$

$$\left\{\left(\frac{1}{g_\beta(\varphi_\beta)}\right)\left[\left(\frac{1}{g_\beta(\varphi_\beta)}\right)\left(\frac{T_\alpha(\varphi_\beta)''}{f_\beta(\varphi_\beta)}\right)' + \left(f_\beta(\varphi_\beta)T_\beta(\varphi_\beta)'\right)\right]' + \frac{T_\beta(\varphi_\beta)''}{f_\beta(\varphi_\beta)} = 0\right\} \quad (16)$$

where

$$f_\alpha(\varphi_\alpha) = \frac{\tau_\alpha(\varphi_\alpha)}{\kappa_\alpha(\varphi_\alpha)}\,,\ g_\alpha(\varphi_\alpha) = \frac{\sigma_\alpha(\varphi_\alpha)}{\kappa_\alpha(\varphi_\alpha)}\,, \varphi_\alpha = \int \kappa_\alpha(s_\alpha),$$

$$f_\beta(\varphi_\beta) = \frac{\tau_\beta(\varphi_\beta)}{\kappa_\beta(\varphi_\beta)},\ g_\beta(\varphi_\beta) = \frac{\sigma_\beta(\varphi_\beta)}{\kappa_\beta(\varphi_\beta)}\,, \varphi_\beta = \int \kappa_\beta(s_\beta)$$

Equation (13) causes

$$f_\alpha(\varphi_\alpha) = f_\beta(\varphi_\beta),\ g_\alpha(\varphi_\alpha) = g_\beta(\varphi_\beta)$$

under the variable transformations $\varphi_\beta = \varphi_\alpha$. So that equations (15) and (16) under the equation (13) and the transformation

$$\varphi_\beta = \int \kappa_\beta(s_\beta) = \int \kappa_\alpha(s_\alpha)$$

are the same. Hence the solution is the same, i.e., the tangent vectors are the same which completes the proof of the theorem.

Theorem: Let $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ be real curves in 4D-Galilean space. Then $\xi_\alpha(s_\alpha)$ and $\xi_\beta(s_\beta)$ are similar curves with variable transformation if and only if the Frenet vectors $E_\alpha(s_\alpha)$ and $E_\beta(s_\beta)$ of curves are the same for all curves

$$E_\alpha(s_\alpha) = E_\beta(s_\beta) \quad (17)$$

under the particular variable transformation

$$\lambda_\alpha^\beta = \frac{ds_\beta}{ds_\alpha} = \frac{\tau_\alpha}{\tau_\beta} = \frac{\sigma_\alpha}{\sigma_\beta} \quad (18)$$

of the arc-lengths.

Proof: Let $\xi_{\alpha}(s_{\alpha})$ and $\xi_{\beta}(s_{\beta})$ be real quaternionic curves with variable transformation. Then from Theorem.4, we have $B(s_{\alpha}) = B(s_{\beta})$. Differentiating this equality with respect to s_{β} gives

$$-\tau_{\beta}(s_{\beta})N_{\beta}(s_{\beta}) + \sigma_{\beta}(s_{\beta})E_{\beta}(s_{\beta}) = \{-\tau_{\alpha}(s_{\alpha})N_{\alpha}(s_{\alpha}) + \sigma_{\alpha}(s_{\alpha})E_{\alpha}(s_{\alpha})\}\frac{ds_{\alpha}}{ds_{\beta}} \quad (19)$$

$$\lambda_{\alpha}^{\beta} = \frac{ds_{\beta}}{ds_{\alpha}} = \frac{\tau_{\alpha}}{\tau_{\beta}} = \frac{\sigma_{\alpha}}{\sigma_{\beta}}$$

Considering equation (19) and (8), we have

$$E_{\alpha}(s_{\alpha}) = E_{\beta}(s_{\beta}) \text{ and } \lambda_{\alpha}^{\beta} = \frac{ds_{\beta}}{ds_{\alpha}} = \frac{\tau_{\alpha}}{\tau_{\beta}} = \frac{\sigma_{\alpha}}{\sigma_{\beta}} \quad (20)$$

Conversely, let $\xi_{\alpha}(s_{\alpha})$ and $\xi_{\beta}(s_{\beta})$ be real quaternionic curves with variable transformation.satisfying (17) and (18). Differentiating (17) with respect to s_{β} it follows

$$\sigma_{\beta}(s_{\beta})B_{\beta}(s_{\beta}) = \frac{ds_{\alpha}}{ds_{\beta}}\sigma_{\alpha}(s_{\alpha})B_{\alpha}(s_{\alpha})$$

From (18) we see that

$$B_{\alpha}(s_{\alpha}) = B_{\beta}(s_{\beta}).$$

Then by theorem 4, we obtain that $\xi_{\alpha}(s_{\alpha})$ and $\xi_{\beta}(s_{\beta})$ are regular similar curves in 4D-Galilean space with variable transformation .

Example: Let $\xi(s) = \left(s, \left(\frac{\sqrt{3}}{2}\right)s, \arctan s - \left(\frac{s}{2}\right), In\sqrt{1+s^2}\right)$ be real curve , $\xi(s)$ forms a family of similar curves in 4D-Galilean space with variable transformations.

Proof: The natural representation of $\xi(s)$ can be written in the form:

$$\xi_{\alpha}(u) = \left(u, \left(\frac{\sqrt{3}}{2}\right)u, \arctan u - \left(\frac{u}{2}\right), In\sqrt{1+u^2}\right). \quad (21)$$

Where $s_{\alpha} = u$ is arclength of the ξ and the curvature is $\kappa_{\alpha}(u)$=1. Differentiating (21), we have

$$\xi'_{\alpha}(u) = \left(1, \frac{\sqrt{3}}{2}, \frac{1}{1+u^2} - \frac{1}{2}, -\frac{u}{1+u^2}\right).$$

Galilean inner product follows that $\langle \xi', \xi' \rangle_{G_4} = 1$. Thus the curve is paremeterized by arclength and tangent vector takes the form:

$$\mathrm{T'}_{\alpha}(u) = \left(1, \frac{\sqrt{3}}{2}, \frac{1}{1+u^2} - \frac{1}{2}, -\frac{u}{1+u^2}\right). \quad (22)$$

From theorem 1, we can write as the following:

$$\gamma_{\beta(s)} = \int \left(1, \left(\frac{\sqrt{3}}{2}\right), \left(\frac{1}{1+(u(s))^2}\right) - \left(\frac{1}{2}\right), -\left(\frac{u(s)}{1+(u(s))^2}\right)\right) ds.$$

where $(s_\beta) = s$. From the equation (18), we have

$$ds_\alpha = \lambda_\beta^\alpha ds_\beta = \left(\frac{\kappa_\beta}{\kappa_\alpha}\right) ds_\beta \quad \text{or} \quad s_\alpha(s_\beta) = \int \left(\frac{\kappa_\beta}{\kappa_\alpha}\right) ds_\beta. \quad (23)$$

If we put the curvature $\kappa_\beta = \kappa(s)$ $(s_\beta = s)$, we have

$$u(s) = \int \kappa(s) ds.$$

The the position vector of ξ with arbitrary curvature $\kappa(s)$ takes the following form:

$$\xi(s) = \int \left(1, \left(\frac{\sqrt{3}}{2}\right), \left(\frac{1}{1+\left(\int \kappa(s)ds\right)^2}\right) - \left(\frac{1}{2}\right), -\left(\frac{\int \kappa(s)ds}{1+\left(\int \kappa(s)ds\right)^2}\right)\right) ds.$$

which is the position vector of ξ . The principal normal vectors of (21) take the form:

$$N(u) = \left(0, 0, -\left(\frac{2}{(1+u^2)^2}\right), \left(\frac{u^2-1}{(1+u^2)^2}\right)\right).$$

Besides, it is easy to write the tangent vector (22) in the simple form:

$$T_{\{\alpha\}(u)} = \int N(u) du = \int (\cosh u, 0, \sinh u, 0) du.$$

From theorem 1, we can write the position vector of a similar curve $\xi_\beta(s) = (\xi_1(s), \xi_2(s), \xi_3(s), \xi_4(s))$ with arbitrary curvature $\kappa(s)$ as follows:

$$\xi_\beta(s) = \int \left\{0, 0, -\frac{2}{\left(1+\left(\int \kappa(s)ds\right)^2\right)^2}, \frac{\left(\int \kappa(s)ds\right)^2 - 1}{\left(1+\left(\int \kappa(s)ds\right)^2\right)^2}\right\} ds. \quad (24)$$

Corollary: The family of real curves in 4D-Galilean space with vanishing principal curvature κ forms a family of real similar curves with variable transformation.

Corollary: The family of real curves in 4D-Galilean space with vanishing torsion τ forms a family of real similar curves with variable transformation.

Corollary: The family of real curves in 4D-Galilean space with vanishing bitorsion σ forms a family of real similar curves with variable transformation.

Conclusion: In 4D-Galilean space, the similar curves are defined and some properties of these curves are obtained. It is shown that real curves with vanishing curvatures form the families of similar curves

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