VERTEX DECOMPOSABILITY PATH COMPLEXES OF DYNKIN GRAPHS

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Abstract. Let G be a simple graph and $\Delta_t(G)$ be a simplicial complex whose facets correspond to the paths of length $t (t \geq 2)$ in G. Let L_n be a line graph on vertices $\{x_1, \dots, x_n\}$ and $\{x_j, y_j\}$ be whisker of L_n at x_j with $3 \leq j \leq n-1$. We give a necessary and sufficient condition that $\Delta_t(L_n \cup W(x_j))$ is vertex decomposable, where $L_n \cup W(x_j)$ is called the graph obtained from L_n by adding a whisker at vertex x_j. As a consequence of our results, vertex decomposability path complexes of Dynkin graphs are shown.

1. Introduction

Dynkin diagrams first appeared in [1] in the connection with classification of simple Lie groups. Among Dynkin diagrams a special role is played by the simply laced Dynkin diagrams A_n, D_n, E_6, E_7 and E_8. Dynkin diagrams are closely related to coxter graphs that appeared in geometry (see [2]).

After that Dynkin diagrams appeared in many branches of mathematics and beyond, em particular em representation theory. In [3] P. Gabriel introduced a notion of a quiver (directed graph) and its representations, He proved the famous Gabriel's theorem on representation of quivers over algebraic closed field. Let Q be a finite quiver and $\bar{Q}$ the undirected graph obtained from Q by deleting the orientation of all arrows.

Theorem 1.1. *(Gabriel theorem). A connected quiver Q is of Finite type if and only if the graph $\bar{Q}$ is one of the following simply laced Dynkin diagrams: A_n, D_n, E_6, E_7 or E_8.*

Let G be an undirected graph over n vertices and t be a fixed integer such that $2 \leq t \leq n$. A sequence $x_{i_1}, \dots, x_{i_t}$ of distinct vertices is called a path of length t if there are $t-1$ distinct edges $e_1, \dots, e_{t-1}$. Where e_j is an edge from x_{i_j} to $x_{i_{j+1}}$.The path complex $\Delta_t(G)$ is defined by $\Delta_t(G) = \langle \{x_{i_1}, \dots, x_{i_t}\} : x_{i_1}, \dots, x_{i_t}$ is a path of length t in $G \rangle$.

In [4] it has been shown that, if G is a directed rooted tree then $\Delta_t(G)$ is a simplicial tree which is sequentially Cohen- Macaulay.

In this work vertex decomposability of $\Delta_t(L_n \cup W(x_j))$ is discussed. We also study vertex decomposability path complexes of Dynkin graphs. In next section, we recall some definitions and results which will be needed later.

In section 3, it is shown that $\Delta_t(L_n \cup W(x_j))$ is vertex decomposable if and only if $2 \leq t \leq 3$ or $\alpha < t \leq n$, where $\alpha = \min\{d(y_j, x_1), d(y_j, x_n)\}$ (see theorem 3-5).

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Also, as an application of our results, vertex decomposability path complexes of Dynkin graphs are shown.

2. Preliminaries

In this section we recall some definitions and results which will be needed later.

Definition 2.1. A simplicial complex Δ over a set of vertices $V = \{x_1, \ldots, x_n\}$, is a collection of subsets of V, with the property that:

(a) $\{x_i\} \in \Delta$, for all i.

(b) If $F \in \Delta$, then all subsets of F are also in Δ (including the empty set) An element in Δ is called a face of Δ.

The dimension of a face F of Δ, dim F, is $|F| - 1$ where, $|F|$ is the number of elements of F.

The maximal faces of Δ. The dimension of the simplicial complex Δ, dim Δ, is the maximum of dimensions of its facets. If all facets of Δ have the same dimension, then Δ is called pure.

Let $\mathcal{F}(\Delta) = \{F_1, \ldots, F_q\}$ be the facet set of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ completely and we write $\Delta = \langle F_1, \ldots, F_q\rangle$.

A simplicial complex with only one facet is called a simplex. A simplicial complex $\Delta^{'}$ is called a subcomplex of Δ, if $F(\Delta^{'}) \subset F(\Delta)$.

For $v \in V$, the subcomplex of Δ obtained by removing all faces $F \in \Delta$ with $v \in F$ is denoted by $\Delta \setminus v$, that is,

$$\Delta \setminus v = \langle F \in \Delta : v \notin F\rangle$$

The link of a face $F \in \Delta$, denoted by $link_{\Delta}F$, is a simplicial complex on V with faces, $G \in \Delta$ such that , $G \cap F = \emptyset$ and $G \cup F \in \Delta$.

Definition 2.2. A simplicial complex Δ is recursively defined to be vertex decomposable, if it is either a simplex, or else has some vertex v such that:

(a) Both $\Delta \setminus v$ and $link_{\Delta}(v)$ are vertex decomposable and (b) no face of $link_{\Delta}(v)$ is a facet of $\Delta \setminus v$.

A vertex v which satisfies in condition (b) is called a shedding vertex.

It is well- known that vertex decomposable $\rightarrow$ Cohen- Macaulay

A simplicial complex Δ is called disconnected, if the vertex set V of Δ is a disjoint union $V = V_1 \cup V_2$ such that no face of Δ has vertices in both V_1 and V_2, otherwise Δ is connected.

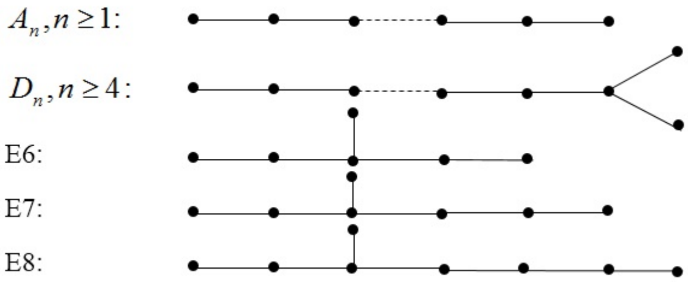
Remark 2.3. All Cohen- Macaulay simplicial complexes of positive dimension are connected.

Let G be a simple graph, and let $T = \{x_1, \ldots, x_n\}$ be a subset of vertices of G. By $G \cup W(T)$ we mean the graph with whiskers $\{y_i, x_i\}$, for each $1 \leq i \leq n$, attached to G.

For simplicity, we shall use $\{\{y_1, x_1\}, \ldots, \{y_n, x_n\}\}$ to denote $W(T)$ in this case.

We use $G \setminus \{x_1, \ldots, x_n\}$ to mean the subgraph obtained from G by removing the vertices $x_1, \ldots, x_n$ and all edges incident to at least one of these vertices.

In the end of this section we define the Dynkin graphs. The following simple graphs are simply laced Dynkin diagrams:



3. Whiskers and vertex decomposability path complexes of lines.

Let L_n be a line graph on vertices $\{x_1, \dots, x_n\}$ and $\{x_j, y_j\}$ be whisker of L_n at x_j with $3 \leq j \leq n-1$.

We obtain some condition that $\Delta_t(L_n \cup W(x_j))$ is vertex decomposable, as an application of our results, vertex decomposability path complexes of Dynkin graphs are shown. Throughout this section we assume $L_n \cup W(x_j)$ be an undirected graph.

Lemma 3.1. *Let $\Delta_t(L_n)$ be a simplicial complex on $\{x_1, \dots, x_n\}$ and $2 \leq t \leq n$. Then $\Delta_t(L_n)$ is vertex decomposable.*

Proof. If $t = n$, then $\Delta_n(L_n)$ is a simplex which is vertex decomposable. Let $2 \leq t < n$ then one has

$$\Delta_t(L_n) = \langle \{x_1, \dots, x_t\}, \{x_2, \dots, x_{t+1}\}, \dots, \{x_{n-t+1}, \dots, x_n\} \rangle.$$

So

$$\Delta_t(L_n) \setminus x_n = \langle \{x_1, \dots, x_t\}, \{x_2, \dots, x_{t+1}\}, \dots, \{x_{n-t}, \dots, x_{n-1}\} \rangle$$

Now we use induction on the number of vertices of L_n and by induction hypothesis $\Delta_t(L_n) \setminus x_n$ is vertex decomposable.

On the other hand, it is clear that $link_{\Delta_t(L_n)}\{x_n\} = \langle \{x_{n-t+1}, \dots, x_{n-1}\} \rangle$.

Thus $link_{\Delta_t(L_n)}\{x_n\}$ is a simplex, which is not a facet of $\Delta_t(L_n) \setminus x_n$. Therefore $\Delta_t(L_n)$ is vertex decomposable. □

Corollary 3.2. *Let A_n be a Dynkin graph and $2 \leq t \leq n$. Then $\Delta_t(A_n)$ is vertex decomposable.*

Proposition 3.3. *Let L_n be a line graph on vertices $\{x_1, \dots, x_n\}$ and $\{x_{n-1}, y_{n-1}\}$ be whisker of L_n at x_{n-1}.*

Then $\Delta_t(L_n \cup W(x_{n-1}))$ is vertex decomposable for all $2 \leq t \leq n$.

Proof. By Lemma 3.1 proof is trivial. □

Corollary 3.4. *Let D_n be a Dynkin graph and $n \geq 4$. Then $\Delta_t(D_n)$ is vertex decomposable for all $2 \leq t \leq n$.*

Proof. We know that $D_n = L_n \cup W(x_{n-1})$. So by proposition 3.3 $\Delta_t(D_n)$ is vertex decomposable. □

Theorem 3.5. *Let L_n be a line graph on vertices $\{x_1, \dots, x_n\}$ and $\{x_j, y_j\}$ be whisker of L_n at x_j with $3 \leq j \leq n-2$.*

Then $\Delta_t(L_n \cup W(x_j))$ is vertex decomposable if and only if $2 \leq t \leq 3$ or $n \geq t > \alpha$, where $\alpha = \min\{d(y_j, x_1), d(y_j, x_n)\}$.

Proof. We first show that $\Delta_t(L_n \cup W(x_j))$ is not vertex decomposable for all $4 \leq t \leq \alpha$. It is well-known that if Δ is a Cohen- Macaulay simplicial complex, then $link_{\Delta}\{F\}$ is Cohen- Macaulay for each face F of Δ.

Also we know that all Cohen- Macaulay complexes of positive dimension are connected. It is easy to see that $link_{\Delta_t(L_n \cup W(x_j))}\{x_j, y_j\}$ is disconnected, and pure of positive dimension.

This implies that $\Delta_t(L_n \cup W(x_j))$ is not Cohen- Macaulay and hence $\Delta_t(L_n \cup W(x_j))$ is not vertex decomposable without loss of generality we can assume that $\alpha = d(y_j, x_1)$ if $t = 2$ or $n \geq t > \alpha$, one has:

$$\Delta_t(L_n \cup W(x_j)) = \langle \{x_1, \ldots, x_t\}, \{x_2, \ldots, x_{t+1}\}, \ldots,$$
$$\{x_{j-1}, x_j, \ldots, x_{j+t-2}\}, \{y_j, x_j, x_{j+1}, \ldots, x_{j+t-2}\}, \ldots, \{x_{n-t+1}, \ldots, x_n\}\rangle$$

So $\Delta_t(L_n \cup W(x_j)) \setminus y_j = \Delta_t(L_n)$ and by lemma 3.1 $\Delta_t(L_n \cup W(x_j)) \setminus y_j$ is vertex decomposable. On the other hand, we have $link_{\Delta_t(L_n \cup W(x_j))}\{y_j\} = \langle \{x_j, x_{j+1}, \ldots, x_{j+t-2}\}\rangle$ which is a simplex and vertex decomposable.

If $t = 3$, then

$$\Delta_3(L_n \cup W(x_j)) = \langle \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \ldots,$$
$$\{x_{j-1}, x_j, y_j\}, \{y_j, x_j, x_{j+1}\}, \ldots, \{x_{n-2}, x_{n-1}, x_n\}\rangle$$

and $\Delta_3(L_n \cup W(x_j)) \setminus y_j = \Delta_3(L_n)$ which is vertex decomposable.

It is easy to see that $link_{\Delta_t(L_n \cup W(x_j))}\{y_j\} = \langle \{x_{j-1}, x_j\}, \{x_j, x_{j+1}\}\rangle$ is vertex decomposable and y_j is a shedding vertex. □

Corollary 3.6. *Let E_6 be a Dynkin graph. Then $\Delta_t(E_6)$ is vertex decomposable if and only if $2 \leq t \leq 3$ or $t = 5$.*

Proof. Since $E_6 = L_5 \cup W(x_3)$, so by theorem 3.5 the proof is completed. □

Corollary 3.7. *Let E_7 be a Dynkin graph. Then $\Delta_t(E_7)$ is vertex decomposable if and only if $2 \leq t \leq 3$ or $5 \leq t \leq 6$.*

Proof. We know that $E_7 = L_6 \cup W(x_3)$ and the proof follow from theorem 3.5. □

Corollary 3.8. *Let E_8 be a Dynkin graph. Then $\Delta_t(E_8)$ is vertex decomposable if and only if $2 \leq t \leq 3$ or $5 \leq t \leq 7$.*

References

[1] E. B. Dynkin, Classification of the simple Lie groups, (Russian) Rec. Math, mat. Spornik (N.S.) 18 (60) (1946), 347-352.
[2] H. S. M. Coxeter, Regular poly to Pes, Third edition. Dover Publications, Inc., New York, (1973).
[3] P. Gabriel, Unzertellungen I, Mansuscr. Math, 6 (1972), 71-103.
[4] J. He and A. Van Tuyl, Algebraic Properties of the Path ideal of tree, Comm. Algebra 38 (2010), 1725-1742.

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