Numerical solution of Kolmogorov equation using compact finite differences method and the cubic spline functions

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Abstract

In this study, we solve the Kolmogorov equation by a compact finite difference method. We apply a compact finite difference approximation for discretizing spatial derivatives. Then, using cubic C^1-spline collocation technique, we solve the time integration of the resulting system of ordinary differential equations. This joined method has fourth-order accuracy in both space and time variables, that is this method is of order $O(h^4, k^4)$. The numerical results confirm the validity of this method.

Mathematics subject classification: 65N06.
Key words: Partial differential equation, Compact method, Cubic C^1-spline collocation method, Kolmogorov equation.

1. Introduction

In probability theory, Kolmogorov equations, including Kolmogorov forward equations and Kolmogorov backward equations are partial differential equations that arise in the theory of continuous-time continuous-state Markov processes t characterize random dynamic processes. In one variable case the Kolmogorov equation is written in the following form

$$\frac{\partial u(x,t)}{\partial t} = \left(-A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2}\right) u(x,t) \qquad (1.1)$$
$$(x,t) \in [a,b] \times [0,T]$$

with initial condition

$$u(x,0) = \varphi(x),$$

and the boundary conditions

$$u(a,t) = \psi_1(t) \ , \quad u(b,t) = \psi_2(t) \ , \quad t \geqslant 0,$$

where $B(x,t) \neq 0$ for all $(x,t) \in [a,b] \times [0,T]$, and $A(x,t)$ and $B(x,t)$ are the continuous and differentiable functions. We assume that ψ_1 and ψ_2 are smooth functions.

The basic approach for high-order compact difference methods is to introduce the standard compact difference approximations to the differential equations and then by repeated differentiation and associated compact differencing, a new high-order compact scheme will be

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developed that incorporates the effect of the leading truncation error terms in the standard method [7]. Recently due to the high-order, compactness and high resolution, we have seen increasing population for high-order compact difference methods in computational fluid dynamics, computational acoustics and electromagnetic [6, 7, 1].

2. Method of solution

In this section we will combine second-order central difference in space with cubic C^1-spline collocation method to obtain a high order method for solving the Kolmogorov equation (1.1). At first we discretize partial differential equation (1.1) in space with central difference to obtain a system of ordinary differential equations with unknown function at each spatial grid point. Then we will apply the cubic C^1-spline collocation method for solving the resulting system of ordinary differential equations. For positive integers n and T, let $h = \frac{b-a}{n}$ denotes the step size of spatial derivatives and k denotes the step size of temporal derivative. So we define

$$x_r = a + rh \quad , \quad r = 0, 1, \cdots, n,$$
$$t_j = jk \quad , \quad j = 0, 1, \cdots .$$

Consider the following partial differential equation

$$f(x) = -A(x,t)\frac{\partial u}{\partial x} + B(x,t)\frac{\partial^2 u}{\partial x^2}. \tag{2.1}$$

If we denote the central difference schemes of order two for second and first derivatives of u as $\delta_x^2 u = \frac{u_{r+1}-2u_r+u_{r-1}}{h^2}$ and $\delta_x u = \frac{u_{r+1}-u_{r-1}}{2h}$, respectively, then we have the following relation for equation (2.1) at point x_r:

$$f_r = -A_r \delta_x u_r + B_r \delta_x^2 u_r - \tau_r, \tag{2.2}$$

in which $B_r = B(x_r, t)$ and $A_r = A(x_r, t)$. The truncation error τ_r is as follows:

$$\tau_r = \frac{h^2}{12} B_r \frac{\partial^4 u}{\partial x^4} - 2A_r \frac{h^2}{12}\frac{\partial^3 u}{\partial x^3} + O(h^4). \tag{2.3}$$

In order to obtain a fourth-order scheme, the fourth and third derivatives of u in (2.3) should be approximated. equation (2.1) gives:

$$\frac{\partial^3 u}{\partial x^3} = \frac{1}{B}\left(\frac{\partial f}{\partial x} + \frac{\partial A}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\left(A - \frac{\partial B}{\partial x}\right)\right). \tag{2.4}$$

Also from (2.4) we have

$$\frac{\partial^4 u}{\partial x^4} = \frac{1}{B}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 A}{\partial x^2}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\left(2\frac{\partial A}{\partial x} - \frac{\partial^2 B}{\partial x^2}\right) + \left(\frac{\partial^3 u}{\partial x^3}\right)\left(A - 2\frac{\partial B}{\partial x}\right)\right). \tag{2.5}$$

By equations (2.2), (2.3), (2.4) and (2.5) for $r = 0, \cdots n-1$ we get

$$\begin{aligned} f_r = -A_r\frac{\partial u}{\partial x} + B_r\frac{\partial^2 u}{\partial x^2} - \frac{h^2}{12}\frac{\partial^2 f}{\partial x^2}\bigg|_r - \frac{h^2}{12B_r}\left(-A - 2\frac{\partial B}{\partial x}\right)_r \frac{\partial f}{\partial x}\bigg|_r \\ -\frac{\partial u}{\partial x}\left(\frac{h^2}{12}\frac{\partial^2 A}{\partial x^2} + \frac{h^2}{12}\left(-A - 2\frac{\partial B}{\partial x}\right)\frac{\partial A}{\partial x}\right)\bigg|_r \\ -\frac{\partial^2 u}{\partial x^2}\left(\frac{h^2}{12}\left(2\frac{\partial A}{\partial x} - \frac{\partial^2 B}{\partial x^2}\right) + \frac{h^2}{12B}\left(-A - 2\frac{\partial B}{\partial x}\right)\left(A - \frac{\partial B}{\partial x}\right)\right)\bigg|_r . \end{aligned} \tag{2.6}$$

Now we rewrite the equation(1.1) for $r = 0, \cdots n-1$ as follows

$$\begin{aligned} f_r + \frac{h^2}{12}\frac{\partial^2 f}{\partial x^2}\Big|_r - \frac{h^2}{12B_r}\left(A + 2\frac{\partial B}{\partial x}\right)_r \frac{\partial f}{\partial x}\Big|_r = \\ -\frac{\partial u}{\partial x}\left(-A - \frac{h^2}{12}\frac{\partial^2 A}{\partial x^2} - \frac{h^2}{12}\left(-A - 2\frac{\partial B}{\partial x}\right)\frac{\partial A}{\partial x}\right)\Big|_r \\ + \frac{\partial^2 u}{\partial x^2}\left(B - \frac{h^2}{12}\left(2\frac{\partial A}{\partial x} - \frac{\partial^2 B}{\partial x^2}\right) - \frac{h^2}{12B}\left(-A - 2\frac{\partial B}{\partial x}\right)\left(A - \frac{\partial B}{\partial x}\right)\right)\Big|_r . \end{aligned} \tag{2.7}$$

Which this relation is a fourth-order compact finite difference scheme for equation (2.1). If we discretize the above equation with second-order central difference in space and each grid point, we obtained the following relation:

$$\begin{aligned} {u_r}' + \frac{h^2}{12}\frac{{u_{r+1}}' - 2{u_r}' + {u_{r-1}}'}{h^2} - \frac{h^2}{12B_r}\left(A - 2\frac{\partial B}{\partial x}\right)_r \frac{{u_{r+1}}' - {u_{r-1}}'}{2h} \\ = \frac{u_{r+1} - u_{r-1}}{2h}P_r^{(1)} + \frac{u_{r+1} - 2u_r + u_{r-1}}{h^2}P_r^{(2)}. \end{aligned} \tag{2.8}$$

In which

$$\begin{aligned} {P_r}^{(1)} = -A_r - \frac{h^2}{12}\frac{\partial^2 A}{\partial x^2}\Big|_r - \frac{h^2}{12}\left(-A - 2\frac{\partial B}{\partial x}\right)_r \frac{\partial A}{\partial x}\Big|_2, \\ {P_r}^{(2)} = B_r - \frac{h^2}{12}\left(2\frac{\partial A}{\partial x} - \frac{\partial^2 B}{\partial x^2}\right)_r - \frac{h^2}{12B_r}\left(-A - 2\frac{\partial B}{\partial x}\right)_r \left(A - \frac{\partial B}{\partial x}\right)_r . \end{aligned}$$

and

$$u_r(t) = u(x_r, t) \quad , \quad {u'}_r(t) = \frac{\partial u}{\partial t}(x_r, t).$$

Then we rewrite the equation (2.8) as follows:

$$\begin{aligned} {u_{r-1}}'\left(\frac{1}{12} + \frac{hA}{24B} + \frac{h}{12B}\frac{\partial B}{\partial x}\right)_r + {u_r}'\left(\frac{5}{6}\right) + {u_{r+1}}'\left(\frac{1}{12} - \frac{hA}{24B} - \frac{h}{12B}\frac{\partial B}{\partial x}\right)_r \\ = u_{r-1}\left(\frac{{P_r}^{(2)}}{h^2} - \frac{{P_r}^{(1)}}{2h}\right) + u_r\left(\frac{-2{P_r}^{(2)}}{h^2}\right) + u_{r+1}\left(\frac{{P_r}^{(2)}}{h^2} + \frac{{P_r}^{(1)}}{2h}\right), \end{aligned} \tag{2.9}$$

If we write (2.9) for each grid point we obtain a system of ordinary differential equations which is as follows:

$$Ru'(t) + c_1(t) = Su(t) + c_2(t). \tag{2.10}$$

in which

$$
\begin{aligned}
&u'(t) = [{u_1}'(t), \ldots, {u_{n-1}}'(t)]^T,\\
&u(t) = [u_1(t), \ldots, u_{n-1}(t)]^T,\\
&R = Trid\left(\frac{1}{12} + \frac{hA_r}{24B_r} + \frac{h}{12B_r}\frac{\partial B}{\partial x}\bigg|_r, \frac{5}{6}, \frac{1}{12} - \frac{hA_r}{24B_r} - \frac{h}{12B_r}\frac{\partial B}{\partial x}\bigg|_r\right)_{(n-1)\times(n-1)},\\
&S = Trid\left(\frac{{P_r}^{(2)}}{h^2} - \frac{{P_r}^{(1)}}{2h}, \frac{-2{P_r}^{(2)}}{h^2}, \frac{{P_r}^{(2)}}{h^2} + \frac{{P_r}^{(1)}}{2h}\right)_{(n-1)\times(n-1)},\\
&c_1(t) = \left[\left(\frac{1}{12} + \frac{hA}{24B} + \frac{h}{12B}\frac{\partial B}{\partial x}\right)_1 {\psi_1}'(t), 0, \ldots, 0, \left(\frac{1}{12} - \frac{hA}{24B} - \frac{h}{12B}\frac{\partial B}{\partial x}\right)_{n-1} {\psi_2}'(t)\right]^T,\\
&c_2(t) = \left[\left(\frac{P^{(2)}}{h^2} - \frac{P^{(1)}}{2h}\right)_1 \psi_1(t), 0, \ldots, 0, \left(\frac{P^{(2)}}{h^2} + \frac{P^{(1)}}{2h}\right)_{n-1} \psi_2(t)\right]^T.
\end{aligned}
$$

If we put $C(t) = c_1(t) - c_2(t)$ and by defining $M = R^{-1}S$ and $P = R^{-1}$ then (2.10) can be written as follows:

$$u'(t) = Mu(t) + PC(t) = F(u(t), t). \tag{2.11}$$

Now we apply the cubic C^1 spline collocation approach [4] to the system of ordinary differential equations (2.11). The cubic C^1 spline collocation method is an A-stable method for solving the first-order ordinary differential equations and has fourth order accuracy (see also[5, 2]).

Let $U(t)$ be a vector that approximates $u(t)$ such that each of its component is a cubic spline function and satisfies in (2.11) at collocation points t_{j-1}, t_j and $t_{j-\frac{1}{2}}$ in the time interval $[t_{j-1}, t_j]$ i.e. $U'(t_l) = F(U(t_l), t_l)$, $l = j-1, j-\frac{1}{2}, j$. From [4] we have the following relations:

$$U(t) = U^{j-1} + kT_1(m){U'}^{j-1} + kT_2(m){U'}^{j-\frac{1}{2}} + kT_3(m){U'}^j, \tag{2.12}$$

where

$$
\begin{aligned}
&T_1(m) = m - \tfrac{3}{2}m^2 + \tfrac{2}{3}m^3, \qquad T_2(m) = 2m^2 - \tfrac{4}{3}m^3,\\
&T_3(m) = -\tfrac{1}{2}m^2 + \tfrac{2}{3}m^3, \qquad t = t_{j-1} + mk, \qquad m \in [0,1]
\end{aligned}
$$

and

$$U^j = U^{j-1} + \frac{k}{6}[MU^{j-1} + PC^{j-1} + 4MU^{j-\frac{1}{2}} + 4PC^{j-\frac{1}{2}} + MU^j + PC^j], \tag{2.13}$$

and

$$U^{j-\frac{1}{2}} = U^{j-1} + \frac{k}{24}[5MU^{j-1} + 5PC^{j-1} + 8MU^{j-\frac{1}{2}} + 8PC^{j-\frac{1}{2}} - MU^j - PC^j], \tag{2.14}$$

in which $U^j = U(t_j), C^j = C(t_j), {U'}^j = U'(t_j)$ and so on. After some manipulation (2.13) and (2.14) can be written as

$$(I - \frac{k}{6}M)U^j = (I - \frac{k}{6}M)U^{j-1} + \frac{2k}{3}MU^{j-\frac{1}{2}} + \frac{k}{6}P(C^{j-1} + 4C^{j-\frac{1}{2}} + C^j), \tag{2.15}$$

Table 3.1: Maximum error obtained for Problem 1 at $T = 1$.

h	Maximum Error
1/5	4.7198×10^{-6}
1/10	6.8234×10^{-7}
1/20	9.0600×10^{-8}
1/40	1.1643×10^{-8}

and

$$(I - \frac{k}{3}M)U^{j-\frac{1}{2}} = (I - \frac{5k}{24}M)U^{j-1} - \frac{k}{24}MU^j + \frac{k}{24}P(5C^{j-1} + 8C^{j-\frac{1}{2}} - C^j), \qquad (2.16)$$

respectively, where I is the $(n-1)\times(n-1)$ identity matrix. Multiplying both sides of (2.15) and (2.16) by $(I - \frac{k}{3}M)$ and $\frac{2k}{3}M$ respectively and adding resulted equations together give as

$$\begin{aligned}(I - \frac{k}{2}M + \frac{k^2}{12}M^2)U^j &= (I + \frac{k}{2}M + \frac{k^2}{12}M^2)U^{j-1} \\ +(\frac{k}{6}P + \frac{k^2}{12}PM)C^{j-1} &+ \frac{2k}{3}PC^{j-\frac{1}{2}} + (\frac{k}{6}P - \frac{k^2}{12}PM)C^j.\end{aligned} \qquad (2.17)$$

So for obtaining the new U^j we should solve a linear system of $(n-1)$ equations and construct approximate solution (2.12) in $[t_{j-1}, t_j]$. Note that by multiplying eqation (2.17) in R^2 we can avoid of any matrix inverting. As we see the amplification matrix, i.e. $(I - \frac{k}{2}M + \frac{k^2}{12}M^2)^{-1}(I + \frac{k}{2}M + \frac{k^2}{12}M^2)$, is the (2,2) Pade approximation of e^{kM}, so the method is fourth-order accurate in time component.

3. Numerical experiments

We applied the methods presented in this article and solved several examples. We performed our computations using **Maple 13** software.

3.1. Test problem 1

Consider equation $\frac{\partial u}{\partial t} = \frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2}$ with $A(x,t) = -1$ and $B(x,t) = 1$. The exact solution is given with

$$u(x,t) = x + t \ , \quad 0 \leqslant x \leqslant 1. \qquad (3.1)$$

The boundary conditions can be obtained easily from exact solution. By applying this technique, equation (3.1) is solved. In Table (3.1) the maximum errors of approximate solutions are shown for $T = 1$ and $h = k$.

3.2. Test problem 2

Consider equation (1.1) with $A(x,t) = 3$ and $B(x,t) = 1$. The exact solution is given with

$$u(x,t) = \frac{1}{\sqrt{1+t}} \exp\left(\frac{-(x-(1+t)A(x,t))^2}{4B(x,t)(1+t)}\right) \ , \quad 0 \leqslant x \leqslant 1 \ . \qquad (3.2)$$

The boundary conditions can be obtained easily from exact solution. The numerical results for $T = 1$ and $h = k$ are shown in Table (3.2).

Table 3.2: Maximum error obtained for Problem 2 at $T = 1$.

h	Maximum Error
1/5	5.3455×10^{-6}
1/10	1.4000×10^{-6}
1/20	2.2516×10^{-7}
1/40	3.1513×10^{-8}

Table 3.3: Error obtained at $T = 1$ for Problem 3.

Grid point	Error
0.1	2.6816×10^{-6}
0.2	3.1583×10^{-6}
0.3	3.5756×10^{-6}
0.4	3.9269×10^{-6}
0.5	4.2102×10^{-6}
0.6	4.4279×10^{-6}
0.7	4.5416×10^{-6}
0.8	4.5709×10^{-6}
0.9	4.4930×10^{-6}

3.3. Test problem 3

Consider equation (1.1) with $A(x, t) = -(x+1)$ and $B(x, t) = 1$. The exact solution is given with

$$u(x, t) = (x + 1)^3 + 8(x + 1)t \ , \quad x \in [0, 1]. \tag{3.3}$$

The boundary conditions and initial condition can be obtained easily from exact solution. By using the introduced methods equation (3.3) is solved. The obtained errors of approximations for $h = \dfrac{1}{20}$ with $T = 1$ are given in Table (3.3).

4. Conclusion

In this paper, we proposed a class of new finite difference schemes, for solving Kolmogorov equation. First we combined a high-order compact finite difference scheme of fourth-order to approximate the spatial derivative with cubic C^1-spline collocation technique, for time integration. This joined method have fourth-order accuracy. The numerical results confirm the validity of this method. In the spline method, we should solve N linear systems of $(n - 1)$ equations. Note that, using spline method in each space step a closed form approximation is obtained.

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