Intuitionistic fuzzy G-Congruences on a Lattice

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Abstract

The definition of reflexivity of a intuitionistic fuzzy relation R on a set S is generalized. Intuitionistic fuzzy equivalence relations on a given set and intuitionistic fuzzy congruence relation on a lattice are studied under this generalized setting. We prove that the set of all intuitionistic fuzzy G-congruence relations on a lattice L forms a lattice, and also we study modularity of a special class of this lattice.

Keywords: Lattice, modular lattice, intuitionistic fuzzy lattice (IFL), intuitionistic fuzzy relation (IFR), Intuitionistic fuzzy equivalence relations, Intuitionistic fuzzy congruence relations.

1. Introduction

The theory of fuzzy sets and fuzzy relations proposed by Zadeh [15,16] in 1965 has achieved a great success in various fields. After that several researchers [1,5,6,9,10,11] have applied the notion of fuzzy sets to congruence .In particular K C Gupta and T P Singh [6] studied fuzzy equivalence relations and fuzzy congruence relations under a generalized setting.

In1986 Atnassov [4] introduced the concept of intuitionistic fuzzy sets which are very effective to deal with vagueness. Later many researchers applied this notion to relations, group theory, ring theory and lattice theory. Recently Hur and his colleagues [7,8] studied intuitionistic fuzzy equivalence relation and intuitionistic fuzzy congruence relation and its properties in lattice theory and group theory. In the studies of intuitionistic fuzzy equivalence relations, the authors are restricted their study by defining intuitionistic fuzzy reflexivity as R(z,z)=(1,0) $\forall z \in S$.

But in [2] N Ajmal & KV Thomas given a different definition of fuzzy reflexivity by defining $\mu_R(z,z)=t$, where $t=\sup_{x,y\in X}\mu_R(x,y)$ when R is an equivalence relation. As a continuation of this work in paper [12] we defined the reflexivity of an intuitionistic fuzzy relation R as R (z, z) = (t, k) $\forall$ z $\in$ X where $t=\sup_{x,y\in X}\mu_R(x,y)$ and k= $\inf_{x,y\in X}\nu_R(x,y)$

In this paper a more generalized version of intuitionistic fuzzy reflexivity called G-reflexive intuitionistic fuzzy relations are defined ,and intuitionistic fuzzy equivalence relations and intuitionistic fuzzy congruence relations under this generalized setting is studied. Also proved that the set of all intuitionistic fuzzy G-congruence relations on a lattice L forms a lattice, and also we study the modularity of a special class of this lattice.

2. Preliminaries

Here we recall some basic definitions related to intuitionistic fuzzy sets and relations, which will be used in the sequel. For details, refer to [4, 8]. Throughout this paper L stands for a lattice (L, $\vee$, $\wedge$).

Definition 2.1. *Let* X *be a non-empty set. An intuitionistic fuzzy set* [IFS] A *of* X *is an object of the following form* A = $\{\angle x, \mu_A(x), \nu_A(x)> | x\in X\}$ *where* $\mu_A: X\rightarrow[01]$ *and* $\nu_A: X\rightarrow[01]$ *define the degree of membership and the degree of non membership of the element* $x\in X$ *respectively, and* $\forall$ $x\in X$, $0\le \mu_A(x)+\nu_A(x)\rangle\le 1$.

Definition 2.2. If A=$\{\langle x,\mu_A(x),\nu_A(x)\rangle / x\in X\}$ and B =$\{\langle x,\mu_B(x),\nu_B(x)\rangle / x\in X\}$ *are any two* IFS of *X*, then

(i) $A\subseteq B\Leftrightarrow \mu_A(x)\le\mu_B(x)$ and $\nu_A(x)\ge\nu_B(x)$ $\forall$ $x\in X$

(ii) A = B $\Leftrightarrow \mu_A(x)=\mu_B(x)$ and $\nu_A(x)=\nu_B(x)$

(iii) $\bar{A}$ = $\{<x, \nu_A(x), \mu_A(x)>/x\in X\}$

(iv) $A^c = =\{x,\mu_A^c(x),\nu_A^c(x)\rangle / x\in X\}$, where $\mu_A^c(x)$ = 1- $\mu_A(x)$ and $\nu_A^c(x)$ = 1 - $\nu_A(x)$,

(v) A $\cap$ B = $\{<x,\min\{\mu_A(x),\mu_B(x)\},\max\{\nu_A(x),\nu_B(x)\}>/x\in X\}$

$= \{< x, \mu_{A\cap B}(x), \nu_{A\cap B}(x) > / x \in X\}$

(vi) $A \cup B = \{< x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\} > / x \in X\}$

$= \{< x, \mu_{A\cup B}(x), \nu_{A\cup B}(x) > / x \in X\}$

Definition 2.3. *Let* L *be a lattice and* A = $\{\langle x, \mu_A(x), \nu_A(x)\rangle / x \in L\}$ *be an IFS of L. Then A is called an **intuitionistic fuzzy sublattice** (IFL) of L if the following conditions are satisfied .*

(i) $\mu_A(x \vee y) \geq \min\{\mu_A(x), \mu_A(y)\}$

(ii) $\mu_A(x \wedge y) \geq \min\{\mu_A(x), \mu_A(y)\}$

(iii) $\nu_A(x \vee y) \leq \max\{\nu_A(x), \nu_A(y)\}$

(iv) $\nu_A(x \wedge y) \leq \max\{\nu_A(x), \nu_A(y)\}$, $\forall\ x, y \in L$

Definition 2.4. *Let* X be *a set. Then a mapping* $R=(\mu_R, \nu_R) : X\times X \to I\times I$ *where* I=[0,1] *is called an **intuitionistic fuzzy relation*** (IFR) *on* X *if* $0 \leq \mu_R(x,y) + \nu_R(x,y) \leq 1$ *for each* $(x,y) \in X\times X$.*That is* $R \in IFS(X\times X)$.

We denote the set of all IFR's *on* X *by* IFR(X).

Definition 2.5. *Let* $R \in IFR(X)$. *Then the inverse of* R *denoted as* R^{-1} *is defined by*

$$R^{-1}(x,y) = R(y,x) , \forall (x,y) \in X\times X.$$

Definition 2.6. *If* R, Q $\in$ IFR(X) *Then their composition* $Q \circ R$ *is defined as*

$$\mu_{Q\circ R}(x,y) = \sup_{z\in X}[\min\{\mu_Q(x,z), \mu_R(z,y)\}]$$

$$\nu_{Q\circ R}(x,y) = \inf_{z\in X}[\max\{\nu_Q(x,z), \nu_R(z,y)\}] .$$

Definition 2.7. *Let* $R \in IFR(X)$.*Then* R *is called an **intuitionistic fuzzy equivalence relation** on* X *if it satisfies the following properties*.

(1) R *is reflexive*, i.e. R(x,x)=(1,0) *for any* $x \in X$

(2) R *is symmetric*, i.e. $R^{-1} = R$

(3) R *is transitive*, i.e. $R \circ R \subseteq R$

The set of all intuitionistic fuzzy equivalence relation on X *is denoted as* IFE(X).

Definition 2.8. Let $R \in IFR(X)$.*Then* R *is called* **(t, k)** ***equivalence relation*** *if*

(1) R *is* (t,k) *reflexive*, i e R(x,x) = (t, k) where $t = \sup_{x,y\in X} \mu_R(x,y)$ and k= $\inf_{x,y\in X} \nu_R(x,y)$.

(2) R *is symmetric*, i.e. $R^{-1} = R$

(3) R *is transitive*, i.e. $R \circ R \subseteq R$.

3. Intuitionistic fuzzy G- equivalence relations

In this section, firstly we modify the present definitions of intuitionistic fuzzy equivalence relation and given a more generalized version, refer to it as intuitionistic fuzzy G-equivalence relation, where G stands for generalized version. Clearly the first two definitions are particular cases of the present one. We also study some properties of these relations under the composition of IFR's.

Definition 3.1. *An* IFR P *on a set* S *is* G-*reflexive if for all* $x \neq y$ *in* S

(1). $\mu_P(x,x) > 0$, $\nu_P(x,x) < 1$

(2) $\mu_P(x,y) \le \delta(P)$ *where* $\delta(P) = \inf_{t\in S} \mu_P(t,t)$

(3) $\nu_P(x,y) \ge \lambda(P)$ *where* $\lambda(P) = \sup_{t\in S} \nu_P(t,t)$

A G-*reflexive*, *transitive* IFR *on* S *is called* G-*preorder on* S.*and symmetric* G-*preorder on* S *is called* G-*equivalence on* S.

Theorem 3.2. *If* P *and* Q *are* G-*reflexive* IFR *on* S *then*

(a) $P \circ Q(x,x) = P \cap Q(x,x)$, $\forall x \in S$

(b) *If* P *is a* G-*preorder on* S *then* $P \circ P = P$.

Proof:(a).Let $x \in S$. Then

$$\mu_{P\circ Q}(x,x) = \sup_{z\in S}[\min\{\mu_P(x,z), \mu_Q(z,x)\}]$$

$$= \min\{\mu_P(x,x), \mu_Q(x,x)\} \text{, since P and Q are G-reflexive}$$

$$= \mu_{P\cap Q}(x,x) \qquad \forall x \in S$$

and

$$\nu_{P\circ Q}(x,x) = \inf_{z\in S}[\max\{\nu_P(x,z), \nu_Q(z,x)\}]$$

$$= \max\{\nu_P(x,x), \nu_Q(x,x)\} \text{, since P and Q are G-reflexive}$$

$$= \nu_{P\cap Q}(x,x) \qquad \forall x \in S$$

Hence $P \circ Q = P \cap Q$.

(b) By transitivity we have $P \circ P \subseteq P$

On the other hand , for x, y $\in$ S, we have

$$\mu_{P\circ P}(x,y) = \operatorname{Sup}_{z\in S}[\min\{\mu_P(x,z), \mu_P(z,y)\}] \geq \min\{\mu_P(x,x), \mu_P(x,y)\} = \mu_P(x,y)$$

and

$$\nu_{P\circ P}(x,y) = \operatorname{Inf}_{z\in S}[\max\{\nu_P(x,z), \nu_P(z,y)\}] \leq \max\{\nu_P(x,x), \nu_P(x,y)\} = \nu_P(x,y)$$

Thus $P \subseteq P \circ P$.Consequently $P \circ P = P$.

Theorem 3.3. *If* P *and* Q *are* G-*equivalence* IFR's *on* S *then so is* $P \cap Q$.

Proof: Since Pand Q are G-reflexive and symmetric $P \cap Q$, is also G-reflexive and symmetric.

Now, for any x,y $\in$ S , we have

$$\begin{aligned}
\mu_{P\cap Q}(x,y) &= \min\{\mu_P(x,y), \mu_Q(x,y)\} \\
&\geq \min\{\sup_{t\in S}\min\{\mu_P(x,t), \mu_P(t,y)\}, \sup_{r\in S}\min\{\mu_Q(x,r), \mu_Q(r,y)\}\} \\
&= \sup_{t\in S}\{\sup_{r\in S}(\min\{\min\{\mu_P(x,t), \mu_P(t,y)\}, \min\{\mu_Q(x,r), \mu_Q(r,y)\}\})\} \\
&\geq \sup_{t\in S}(\min\{\min\{\mu_P(x,t), \mu_P(t,y)\}, \min\{\mu_Q(x,t), \mu_Q(t,y)\}\}) \\
&= \sup_{t\in S}(\min\{\min\{\mu_P(x,t), \mu_Q(x,t)\}, \min\{\mu_P(t,y), \mu_Q(t,y)\}\}) \\
&= \sup_{t\in S}\min\{\mu_{P\cap Q}(x,t), \mu_{P\cap Q}(t,y)\} \\
&= \mu_{(P\cap Q)\circ(P\cap Q)}(x,y)
\end{aligned}$$

$$\begin{aligned}
\nu_{P\cap Q}(x,y) &= \max\{\nu_P(x,y), \nu_Q(x,y)\} \\
&\leq \max\{\inf_{t\in S}\max\{\nu_P(x,t), \nu_P(t,y)\}, \inf_{r\in S}\max\{\nu_Q(x,r), \nu_Q(r,y)\}\} \\
&= \inf_{t\in S}\{\inf_{r\in S}(\max\{\max\{\nu_P(x,t), \nu_P(t,y)\}, \max\{\nu_Q(x,r), \nu_Q(r,y)\}\})\} \\
&\leq \inf_{t\in S}(\max\{\max\{\nu_P(x,t), \nu_P(t,y)\}, \max\{\nu_Q(x,t), \nu_Q(t,y)\}\}) \\
&= \inf_{t\in S}(\max\{\max\{\nu_P(x,t), \nu_Q(x,t)\}, \max\{\nu_P(t,y), \nu_Q(t,y)\}\}) \\
&= \inf_{t\in S}\max\{\nu_{P\cap Q}(x,t), \nu_{P\cap Q}(t,y)\} \\
&= \nu_{(P\cap Q)\circ(P\cap Q)}(x,y)
\end{aligned}$$

Hence $(P \cap Q) \circ (P \cap Q) \subseteq (P \cap Q)$.

Thus $(P \cap Q)$ is transitive on S.

Result 3.4. *The following example shows if* P*and* Q *are* G*-reflexive intuitionistic fuzzy relations on* S *then* $P \circ Q$ *may not be* G*-reflexive.*

Let S={a,b}.Define intuitionistic fuzzy relations Pand Q as

P(a, a)=(1/2,1/2) P(b,b)=(1/3,1/4) P(a,b)= (1/4,3/4) P(b, a)=(1/5,3/5)

and

Q(a, a)=(1/2,1/3) Q(b,b)=(1/2,1/2) Q(a, b)= (1/2,1/2) Q(b,a) =(1/4,3/4)

It can easily verify that P and Q are G- reflexive

But $\underset{P\circ Q}{\mu}(a,b)$ =1/2 > $\underset{P\circ Q}{\mu}(b,b)$=1/3

Hence $P \circ Q$ is not G-reflexive on S.

Theorem 3.5. *Let* P *and* Q *be* G*-reflexive intuitionistic fuzzy relations on* S *such that*

$\max\{\underset{P}{\mu}(x,y), \underset{Q}{\mu}(x,y)\} \le \min\{\delta(P),\delta(Q)\}$ *and* $\min\{\underset{P}{\nu}(x,y),\underset{Q}{\nu}(x,y)\} \ge \max\{\lambda(P),\lambda(Q)\}$,

$\forall\, x \ne y$ in S.*Then* $P \circ Q$ *is* G*-reflexive on* S *with* $\delta(P \circ Q) = \min\{\delta(P),\delta(Q)\}$.

Proof: Firstly note that

$$\delta(P\circ Q) = \underset{x\in S}{Inf}\ \underset{P\circ Q}{\mu}(x,x)$$

$$= \underset{x\in S}{Inf}\ [\min\{\underset{P}{\mu}(x,x), \underset{Q}{\mu}(x,x)\}] \text{ [By Theorem3.2 } P\circ Q(x,x) = P\cap Q(x,x)]$$

$$= \min\{\underset{x\in S}{Inf}\ \underset{P}{\mu}(x,x), \underset{x\in S}{Inf}\ \underset{Q}{\mu}(x,x)\}$$

$$= \min\{\delta(P),\delta(Q)\}$$

and

$$\lambda(P\circ Q) = \sup_{x\in S} \underset{P\circ Q}{\nu}(x,x)$$

$$= \sup_{x\in S} [\max\{\underset{P}{\nu}(x,x), \underset{Q}{\nu}(x,x)\}]$$

$$= \max\{\sup_{x\in S} \underset{P}{\nu}(x,x), \sup_{x\in S} \underset{Q}{\nu}(x,x)\}$$

$$= \max\{\lambda(P),\lambda(Q)\}$$

Since P and Q are G- reflexive, $\forall\, x \in S$

$$\underset{P\circ Q}{\mu}(x,x) = \min\{\underset{P}{\mu}(x,x), \underset{Q}{\mu}(x,x)\} > 0$$

and (1)

$$\nu_{P\circ Q}(x,x) = \max\{\nu_P(x,x), \nu_Q(x,x)\} < 1$$

Let $x \neq y \in S$. If $x \neq t$ and $t \neq y$, then

$$\min\{\mu_P(x,t), \mu_Q(t,y)\} \leq \min\{\delta(P), \delta(Q)\} \text{ since P and Q are G-reflexive.}$$

On the other hand, if x = t,then

$$\min\{\mu_P(x,x), \mu_Q(x,y)\} \leq \mu_Q(x,y) \leq \max\{\mu_P(x,y), \mu_Q(x,y)\}$$

$$\leq \min\{\delta(P), \delta(Q)\} \quad \text{(By the given hypothesis)}$$

and if y = t, then

$$\min\{\mu_P(x,y), \mu_Q(y,y)\} \leq \mu_P(x,y) \leq \max\{\mu_P(x,y), \mu_Q(x,y)\} \leq \min\{\delta(P), \delta(Q)\}$$

Therefore

$$\mu_{P\circ Q}(x,y) \leq \min\{\delta(P), \delta(Q)\} = \delta(P \circ Q) \quad . \tag{2}$$

Next, let $x \neq y \in S$. If $x \neq t$ and $t \neq y$ Then we have

$$\max\{\nu_P(x,t), \nu_Q(t,y)\} \geq \max\{\lambda(P), \lambda(Q)\}$$

Also,for x = t we have

$$\max\{\nu_P(x,x), \nu_Q(x,y)\} \geq \nu_Q(x,y) \geq \min\{\nu_P(x,y), \nu_Q(x,y)\}$$

$$\geq \max\{\lambda(P), \lambda(Q)\} \quad \text{(By given hypothesis)}$$

And y = t we have

$$\max\{\nu_P(x,y), \nu_Q(y,y)\} \geq \nu_P(x,y) \geq \min\{\nu_P(x,y), \nu_Q(x,y)\} \geq \max\{\lambda(P), \lambda(Q)\}$$

Therefore

$$\nu_{P\circ Q}(x,y) \geq \max\{\lambda(P), \lambda(Q)\} = \lambda(P \circ Q) \tag{3}$$

Hence from (1), (2) and (3) $P \circ Q$ is G-reflexive.

Corollary 3.6. *If* P*and* Q *are* G-*reflexive intuitionistic fuzzy relations on* S *with* $\delta(P) = \delta(Q)$ and $\lambda(P)=\lambda(Q)$ *then* $P \circ Q$ *is* G-*reflexive with*

$$\delta(P \circ Q) = \delta(P) = \delta(Q) \text{ and } \lambda(P \circ Q) = \lambda(P) = \lambda(Q)$$

4. Intuitionistic fuzzy G-congruence relations on a lattice

Here we study intuitionistic fuzzy congruence relations on a lattice L under the generalized setting which we call as intuitionistic fuzzy G-congruence relation. At the end of this section we prove that the set of all G-congruence relations on L forms a lattice, and we study the modularity of a special class of this lattice.

Definition 4.1. *An intuitionistic fuzzy G-equivalence relation* P *on* L *is called a G-ongruence relation if it satisfies the following compatibility relations.*

$$\mu_P(x,y)\wedge\mu_P(z,t)\le\mu_P(x\vee z,y\vee t)$$
$$\mu_P(x,y)\wedge\mu_P(z,t)\le\mu_P(x\wedge z,y\wedge t)$$
$$\nu_P(x,y)\vee\nu_P(z,t)\ge\nu_P(x\vee z,y\vee t)$$
$$\nu_P(x,y)\vee\nu_P(z,t)\ge\nu_P(x\wedge z,y\wedge t)\qquad\forall\ x,y,z,t\in L$$

Example 4.2. *Consider lattice* L ={1,2,5,10} *under divisibility. We define an* IFR P *on* L *by,*

P(x, y) = (1/4, 3/4) (if x $\neq$ y) P(1,1) =(1,0),P(2,2)= (5,5) =(1/3,3/5),and P(10,10)=(1/2,1/2).It can easily verify that P is a G-congruence relation on L.

Theorem 4.3. *If* P *and* Q *are G-congruence intuitionistic fuzzy relations on* L *then* P$\cap$Q *is a G-congruence relation on* L.

Proof: FromTheorem3.2,P$\cap$Q is G-equivalence relation on L.We show that it is compatible. For this ,let x,y,z,t $\in$ L,Then

$$\mu_{P\cap Q}(x\vee z,y\vee t)=\min\{\mu_P(x\vee z,y\vee t),\mu_Q(x\vee z,y\vee t)\}$$
$$\ge\min\{\min\{\mu_P(x,y),\mu_P(z,t)\},\min\{\mu_Q(x,y),\mu_Q(z,t)\}\},$$

Since P and Q are congruence relations

$$=\min\{\min\{\mu_P(x,y),\mu_Q(x,y)\},\min\{\mu_P(z,t),\mu_Q(z,t)\}\}$$
$$=\min\{\mu_{P\cap Q}(x,y),\mu_{P\cap Q}(z,t)\}$$

Similarly

$$\mu_{P\cap Q}(x\wedge z,y\wedge t)\ge\min\{\mu_{P\cap Q}(x,y),\mu_{P\cap Q}(z,t)\}$$

Also,we have

$$\nu_{P\cap Q}(x\vee z, y\vee t) = \max\{\nu_P(x\vee z, y\vee t), \nu_Q(x\vee z, y\vee t)\}$$
$$\leq \max\{\max\{\nu_P(x,y), \nu_P(z,t)\}, \max\{\nu_Q(x,y), \nu_Q(z,t)\}\}$$
$$= \max\{\max\{\nu_P(x,y), \nu_Q(x,y)\}, \max\{\nu_P(z,t), \nu_Q(z,t)\}\}$$
$$= \max\{\nu_{P\cap Q}(x,y), \nu_{P\cap Q}(z,t)\}$$

Similarly

$$\nu_{P\cap Q}(x\wedge z, y\wedge t) \leq \max\{\nu_{P\cap Q}(x,y), \nu_{P\cap Q}(z,t)\}$$

Hence P$\bigcap$Q is a G- congruence relation on L.

Theorem 4.4. *Let* P *and* Q *be G-congruence relations on the lattice* L *suchthat*P$\circ$Q=Q$\circ$P *and* $\delta(P) = \delta(Q)$ and λ(P)=λ(Q). *Then* P$\circ$Q *is also a G-congruence on* L *with* $\delta(P\circ Q) = \delta(P) = \delta(Q)$ and $\lambda(P\circ Q) = \lambda(P) = \lambda(Q)$

Proof: Wehave P$\circ$Q is G-reflexive

with $\delta(P\circ Q) = \delta(P) = \delta(Q)$ and $\lambda(P\circ Q) = \lambda(P) = \lambda(Q)$ (By corrollary3.5)

Next, we have

$$\mu_{P\circ Q}(x,y) = \operatorname*{Sup}_{z\in L}[\min\{\mu_P(x,z), \mu_Q(z,y)\}]$$

$$= \operatorname*{Sup}_{z\in L}[\min\{\mu_P(z,x), \mu_Q(y,z)\}]$$, since Pand Q are symmetric.

$$= \operatorname*{Sup}_{z\in L}[\min\{\mu_Q(y,z), \mu_P(z,x)\}]$$

$$= \mu_{Q\circ P}(y,x)$$

$$= \mu_{P\circ Q}(y,x),$$ since P$\circ$Q=Q$\circ$P

Similarly

$$\nu_{P\circ Q}(x,y) = \operatorname*{Inf}_{z\in L}[\max\{\nu_P(x,z), \nu_Q(z,y)\}]$$

$$= \operatorname*{Inf}_{z\in L}[\max\{\nu_P(z,x), \nu_Q(y,z)\}],$$ since Pand Q are symmetric.

$$= \operatorname*{Inf}_{z\in L}[\max\{\nu_Q(y,z), \nu_P(z,x)\}]$$

$$= \nu_{Q\circ P}(y,x)$$

$$= \nu_{P\circ Q}(y,x) \text{ since } P\circ Q=Q\circ P$$

Hence P ∘ Q is symmetric.

Since intuitionistic fuzzy relations satisfy associative property, we have

$$(P\circ Q)\circ(P\circ Q) = P\circ(Q\circ P)\circ Q = P\circ(P\circ Q)\circ Q = (P\circ P)\circ(Q\circ Q) \subseteq P\circ Q$$

Hence P ∘ Q is transitive

Finally we prove that P ∘ Q is compatible.

Let x,y,z,t ∈ L,Then

$$\mu_{P\circ Q}(x\vee y, z\vee t) = \sup_{k\in L}[\min\{\mu_P(x\vee y,k), \mu_Q(k,z\vee t)\}]$$

$$\geq \sup_{r\in L}[\sup_{s\in L}[\min\{\mu_P(x\vee y, r\vee s), \mu_Q(r\vee s, z\vee t)\}]]$$

$$\geq \sup_{r\in L}[\sup_{s\in L}[\min\{\min[\mu_P(x,r),\mu_P(y,s)],\min[\mu_Q(r,z),\mu_Q(s,t)]\}]] \text{ ,since P \& Q are compactible}$$

$$= \sup_{r\in L}[\sup_{s\in L}[\min\{\min[\mu_P(x,r),\mu_Q(r,z)],\min[\mu_P(y,s),\mu_Q(s,t)]\}]]$$

$$= \min\{\sup_{r\in L}(\min[\mu_P(x,r),\mu_Q(r,z)]), \sup_{s\in L}(\min[\mu_P(y,s),\mu_Q(s,t)])\}$$

$$= \min\{\mu_{P\circ Q}(x,z), \mu_{P\circ Q}(y,t)\}$$

Similarly ,we get

$$\mu_{P\circ Q}(x\wedge y, z\wedge t) \geq \min\{\mu_{P\circ Q}(x,z), \mu_{P\circ Q}(y,t)\}$$

Also, we have

$$\nu_{P\circ Q}(x\vee y, z\vee t) = \inf_{k\in L}[\max\{\nu_P(x\vee y,k), \nu_Q(k,z\vee t)\}]$$

$$\leq \inf_{r\in L}[\inf_{s\in L}[\max\{\nu_P(x\vee y, r\vee s), \nu_Q(r\vee s, z\vee t)\}]]$$

$$\leq \inf_{r\in L}[\mathit{Inf}_{s\in L}[\max\{\max[\nu_P(x,r),\nu_P(y,s)],\max[\nu_Q(r,z),\nu_Q(s,t)]\}]] \text{ ,since P \& Q are compatible.}$$

$$= \inf_{r\in L}[\mathit{Inf}_{s\in L}[\max\{\max[\nu_P(x,r),\nu_Q(r,z)],\max[\nu_P(y,s),\nu_Q(s,t)]\}]]$$

$$= \max\{\mathit{Inf}_{r\in L}(\max[\nu_P(x,r),\nu_Q(r,z)]), \inf_{s\in L}(\max[\nu_P(y,s),\nu_Q(s,t)])\}$$

$$=\max\{\underset{P\circ Q}{\nu}(x,z),\ \underset{P\circ Q}{\nu}(y,t)\}$$

Similarly, we can obtain

$$\underset{P\circ Q}{\nu}(x\wedge y, z\wedge t)\leq\ \max\{\underset{P\circ Q}{\nu}(x,z),\ \underset{P\circ Q}{\nu}(y,t)\}$$

Hence P $\circ$ Q is a G-congruence relation on L

Theorem 4.5. *If P and Q are intuitionistic fuzzy G-congruence relations on L such that,* $\delta(P)=\delta(Q)$, $\lambda(P)=\lambda(Q)$ and their *composition* $P\circ Q$ *is also an intuitionistic fuzzy G-congruence relation on* L, *Then* $P\circ Q=P\vee Q$ *where* $P\vee Q$ *is the least upper bound for P, Q with respest to intuitionistic fuzzy set inclusion.*

Proof.

For any a,b $\in$ L, we have

$$\underset{P\circ Q}{\mu}(a,b)=\underset{z\in L}{Sup}[\min\{\underset{P}{\mu}(a,z),\underset{Q}{\mu}(z,b)\}]$$

$$\geq\min\{\underset{P}{\mu}(a,b)\wedge\underset{Q}{\mu}(b,b)\}$$

$$=\underset{P}{\mu}(a,b)\ \text{, since } \underset{P}{\mu}(a,b)\leq\delta(P)=\delta(Q)\leq\underset{Q}{\mu}(b,b)$$

Similarly, we have $\underset{P\circ Q}{\mu}(a,b)\geq\underset{Q}{\mu}(a,b)$

Also

$$\underset{P\circ Q}{\nu}(a,b)=\underset{z\in L}{Inf}[\max\{\underset{P}{\nu}(a,z),\underset{Q}{\nu}(z,b)\}]$$

$$\leq\max\{\underset{P}{\nu}(a,b),\underset{Q}{\nu}(b,b)\}$$

$$=\underset{P}{\nu}(a,b)\quad\text{,since } \underset{P}{\nu}(a,b)\geq\lambda(P)=\lambda(Q)\geq\underset{Q}{\nu}(b,b)$$

Smilarly,we get $\underset{P\circ Q}{\nu}(a,b)\leq\underset{Q}{\nu}(a,b)$

Hence P $\circ$ Q $\supseteq$ P and P $\circ$ Q $\supseteq$ Q

Now let R be an intuitionistic fuzzy G-congruence relation on L such that R $\supseteq$ P and R $\supseteq$ Q

Then for any a,b $\in$ L we have

$$\underset{P\circ Q}{\mu}(a,b)=\underset{z\in L}{Sup}[\min\{\underset{P}{\mu}(a,z),\underset{Q}{\mu}(z,b)\}]$$

$$\leq\underset{z\in L}{Sup}[\min\{\underset{R}{\mu}(a,z),\underset{R}{\mu}(z,b)\}]$$

$$= \mu_{R\circ R}(a,b)$$

$$\leq \mu_R(a,b) \quad \text{, since } R\circ R \subseteq R$$

and

$$\nu_{P\circ Q}(a,b) = \inf_{z\in L}[\max\{\nu_P(a,z), \nu_Q(z,b)\}]$$

$$\geq \inf_{z\in L}[\max\{\nu_R(a,z), \nu_R(z,b)\}]$$

$$= \nu_{R\circ R}(a,b)$$

$$\geq \nu_R(a,b) \quad \text{, since } R\circ R \subseteq R$$

Hence P ∘ Q ⊆ R. Thus we have P ∘ Q is the least upper bound of P and Q

That is P ∘ Q= P ∨ Q.

Let us denote the class of all intuitionistic fuzzy G-congruence relations on L by $IFC_G(L)$.

Then $IFC_G(L)$ *forms a lattice under the operations* ∧ *and* ∨ *where* $P\wedge Q = P\cap Q$ *and*

$P\vee Q$ *is the intuitionistic fuzzy G-congruence relation generated by both P and Q.*

Theorem 4.6. *Let* $\mathscr{A}$ *any sublattice of* $(IFC_G(L), \subseteq, \cap, \vee)$ *such that* $\delta(P)=\delta(Q)$, $\lambda(P)=\lambda(Q)$ *and* P ∘ Q=Q ∘ P *for any* P,Q ∈ $\mathscr{A}$. *Then* $\mathscr{A}$ *is a modular lattice.*

Proof: Let P,Q,R ∈ $\mathscr{A}$ such that P ⊆ R

Then we have to show that (P ∨ Q) ∩ R ⊆ P ∨ (Q ∩ R)

Let x,y ∈ S.Then

$$\mu_{(P\vee Q)\cap R}(x,y) = \mu_{(P\circ Q)\cap R}(x,y) \quad \text{since } P\circ Q = P\vee Q$$

$$= \min\{\sup_{z\in L}[\min\{\mu_P(x,z), \mu_Q(z,y)\}], \mu_R(x,y)\}$$

$$= \sup_{z\in L}[\min\{\mu_P(x,z), \mu_Q(z,y), \mu_P(x,z), \mu_R(x,y)\}]$$

$$\leq \sup_{z\in L}[\min\{\mu_P(x,z), \mu_Q(z,y), \mu_R(x,z), \mu_R(x,y)\}] \quad \text{,since } P\subseteq R$$

$$\leq \sup_{z\in L}[\min\{\mu_P(x,z), \mu_Q(z,y), \mu_R(z,y)\}], \quad \text{since } R\in IFC_G(L)$$

$$= \mu_{P\circ(Q\cap R)}(x,y)$$

$$= \mu_{P\vee(Q\cap R)}(x,y)$$

and

$$\nu_{(P\vee Q)\cap R}(x,y) = \nu_{(P\circ Q)\cap R}(x,y) \text{ since P} \circ \text{ Q=P } \vee \text{Q}$$

$$= \max\{\inf_{z\in L}[\max\{\nu_P(x,z),\nu_Q(z,y)\}],\nu_R(x,y)\}$$

$$= \inf_{z\in L}[\max\{\nu_P(x,z),\nu_Q(z,y),\nu_P(x,z),\nu_R(x,y)\}]$$

$$\geq \inf_{z\in L}[\max\{\nu_P(x,z),\nu_Q(z,y),\nu_R(x,z),\nu_R(x,y)\}] \text{, since P} \subseteq \text{R}$$

$$\geq \inf_{z\in L}[maz\{\nu_P(x,z),\nu_Q(z,y),\nu_R(z,y)\}] \text{ , since R} \in IFC_G(L)$$

$$= \nu_{P\circ(Q\cap R)}(x,y)$$

$$= \nu_{P\vee(Q\cap R)}(x,y)$$

Hence (P$\vee$ Q)$\cap$ R $\subseteq$ P$\vee$(Q$\cap$R)

Thus $\mathscr{A}$ is modular sublattice of the lattice ($IFC_G(L)$,$\subseteq,\cap,\vee$) .

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