Some Star related *I*-cordial graphs

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Abstract

An *I*-cordial labeling of a graph G = (V, E) is an injective map f from V to $\left[-\frac{p}{2}..\frac{p}{2}\right]^*$ or $\left[-\left\lfloor\frac{p}{2}\right\rfloor..\left\lfloor\frac{p}{2}\right\rfloor\right]$ as p is even or odd, respectively be an injective mapping such that f(u) + f(v) ≠ 0 and induces an edge labeling $f^*: E \rightarrow \{0, 1\}$ where, $f^*(uv) = 1$ if f(u) + f(v) > 0 and $f^*(uv) = 0$ otherwise, such that the number of edges labeled with1and the number of edges labeled with 0 differ at most by 1. If a graph has *I*-cordial labeling, then it is called ***I*-cordial graph**.In this paper, we prove that $B_{m,n}$, $S'(B_{n,n})$, $D_2(B_{m,n})$ are *I*-cordial; $K_{n,n}$ is *I*-cordial only if n is even ; $K_{m,n}$ is *I*-cordial only if m or n is even and $B_{n,n}^2$ is not *I*-cordial.

Notation: Here $[-x..x] = \{t / t$ is an integer and $|t| \leq x\}$ and $[-x..x]^* = [-x..x] - \{0\}$.

Key Words: Cordial labeling; *I*-cordial labeling.

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1. INTRODUCTION

By a graph we mean a finite undirected graph without loops and multiple edges. For terms not defined here we refer to Harary [9].

An *I*-cordial labeling of a graph G (V, E) is an injective map f from V to $\left[-\frac{p}{2}..\frac{p}{2}\right]^*$ or $\left[-\left\lfloor\frac{p}{2}\right\rfloor..\left\lfloor\frac{p}{2}\right\rfloor\right]$ as p is even or odd, respectively be an injective mapping such that $f(u) + f(v) \neq 0$ and induces an edge labeling $f^*: E \rightarrow \{0, 1\}$ where, $f^*(uv) = 1$ if $f(u) + f(v) > 0$ and $f^*(uv) = 0$ otherwise, such that the number of edges labeled with 1 and the number of edges labeled with 0 differ at most by 1. If a graph has *I*-cordial labeling, then it is called ***I*-cordial graph**. The concept of cordial graph originated from I.Cahit [1,2] in 1987 as a weaker version of graceful and harmonious graphs and was based on {0,1} binary labeling of vertices.

Let $f: V \rightarrow \{0, 1\}$ be a mapping that induces an edge labeling $\overline{f}: E \rightarrow \{0, 1\}$ defined by $\overline{f}(uv) = |f(u) - f(v)|$. Cahit called such a labeling cordial if the following condition is satisfied: $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(i)$ and $e_f(i)$, $i = 0, 1$ are the number of vertices and edges of G respectively with label i (under f and $\overline{f}$ respectively). A graph G is called cordial if it admits cordial labeling.

In [1], Cahit showed that (i) every tree is cordial (ii) K_n is cordial if and only if $n \leq 3$ (iii) $K_{r,s}$ is cordial for all r and s (iv) the wheel W_n is cordial if and only if $n \equiv 3(\text{mod}4)$ (v) C_n is cordial if and only if $n \not\equiv 2(\text{mod}4)$ (vi) an Eulerian graph is not cordial if its size is congruent to 2 modulo 4.

Du [4] investigated cordial complete k-partite graphs. Kuo et al. [13] determined all m and n for which mK_n is cordial. Lee et al. [14] exhibited some cordial graphs. Generalised Peterson graphs that are cordial are characterised in [7]. Ho et.al [6] investigated the construction of cordial graphs using Cartesian products and composition of graphs. Shee and Ho [7] determined the cordiality of $C_m^{(n)}$; the one–point union of n copies of C_m. Several constructions of cordial graphs were proposed in [10-12, 15-19]. Other results and open problems concerning cordial graph are seen in [2, 5]. Other types of cordial graphs were considered in [3, 4, 8, 21]. Vaidya et.al [22] has also discussed the cordiality of various graphs.

Definition 1.1 [24]

Let f be a binary edge labeling of graph G = {V, E}and the induced vertex labeling is given by $f(v) = \sum_{\forall u} f(u, v) (\text{mod}2)$ where $v \in V$ and $\{u,v\} \in E$. f is called an **E-cordial labeling** of G if $|e_f(0) - e_f(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$, where $e_f(0)$ and $e_f(1)$ denote the number of edges, and $v_f(0)$ and $v_f(1)$ denote the number of vertices with 0's and 1's respectively. The graph G is called **E-cordial** if it admits E-cordial labeling.

In 1997 Yilmaz and Cahit [24] have introduced E-cordial labeling as a weaker version of edge–graceful labeling. They proved that the trees with n vertices, K_n, C_n are E-cordial if and only if $n \not\equiv 2$ (mod4) while $K_{m,n}$ admits E-cordial labeling if and only if $m + n \not\equiv 2(\text{mod}4)$.

Definition 1.2 [21]

A **prime cordial labeling** of a graph G with vertex set V is a bijection f from V to {1, 2, 3, . . . , |V|} where each edge uv is assigned the label 1 if gcd (f(u), f(v)) = 1 and 0 if gcd (f(u), f(v)) > 1, such that the number of edges having label 0 and edges having label 1differ by at most 1.

Sundaram et.al. [20] introduced the notion of prime cordial labeling. They proved the following graphs are prime cordial: C_n if and only if $n \geq 6$; P_n if and only if $n \neq 3$ or 5; $K_{1,n}$(n, odd); the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \geq 3$; bi-stars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders. J. Babujee and L.Shobana [23] proved the existence of prime cordial labeling for sun graph, kite graph and coconut tree and Y-tree, $< K_{1,n}: 2 >$ ($n \geq 1$); Hoffman tree, and $K_2 \Theta C_n$ (C_n)

In this paper, we prove that $B_{m,n}$, $S'(B_{n,n})$, $D_2(B_{m,n})$ are *I*-cordial; $K_{n,n}$ is *I*-cordial only if n is even ; $K_{m,n}$ is *I*-cordial only if m or n is even and $B_{n,n}^2$ is not *I*-cordial

Notation. 1.3

(i) $[-x..x] = \{t \,/\, t$ is an integer and $|t| \leq x\}$

(ii) $[-x..x]^* = [-x..x] - \{0\}$

2. Main Results

3. Definition.2.1

Let G = (V,E) be a simple connected graph with p vertices. Let $f: V \rightarrow \left[-\frac{p}{2}..\frac{p}{2}\right]^*$ or $\left[-\left\lfloor\frac{p}{2}\right\rfloor..\left\lfloor\frac{p}{2}\right\rfloor\right]$ as p is even or odd respectively be an injective mapping such that $f(u) + f(v) \neq 0$ and induces an edge labeling $f^*: E \rightarrow \{0, 1\}$ such that $f^*(uv) = 1$, if $f(u) + f(v)$

> 0 and f(uv) = 0 otherwise. Let $e_f(i)$ = number of edges labeled with i, where i = 0 or 1. f is said to be ***I*-cordial** if $|e_f(0) - e_f(1)| \leq 1$. A graph G is called *I*-cordial if it admits a ***I*-cordial labeling**.

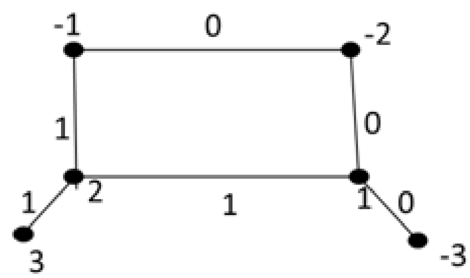

Fig 1. *I*-cordial Graph

Theorem 2.2[16] Let G be a (p, q) graph where p is even. Then G is *I*-cordial only if d(v) < p – 1, for any vertex v ϵ V(G).

Theorem 2.3[16] Let G be a (p, q) graph where p is odd. If there exist two vertices u and v such that d(u) = p – 1 = d(v). Then G is not an *I*-cordial.

Theorem 2.4 $B_{m,n}$ is I-cordial.

Proof. $B_{m,n}$ has (m + n + 2) vertices and (m + n + 1) edges. Let u be the vertices of degree m and v be the vertices of degree n, such that uv is a common edge in $B_{m,n}$. Let $u_1, u_2, \ldots, u_m$ denote the vertices adjacent to u and $v_1, v_2, \ldots, v_n$ denote the vertices adjacent to u in $B_{m,n}$. Then $E(B_{m,n}) = \{uv\} \cup A \cup B$, where $A = \{uu_i\}_{i=1}^{m}$ and $B = \{uv_i\}_{i=1}^{n}$.

CASE1. Both m and n are even or m and n are odd.

Let $f : V \rightarrow \left[\frac{-(m+n+2)}{2} \,..\, \frac{(m+n+2)}{2}\right]^*$ be a mapping defined by f(u) = -1 and f(v) = 2, $f(u_1) = -2$, $f(v_1) = 1$. The other vertices $\{u_2, u_3, \ldots, u_m, v_2, \ldots, v_n\}$ can be arranged with the labels,

{-(n+1), . . . , -4, -3, 3, 4, . . . , (n + 1)} in such a way that the vertices are given positive and negative labels. Since, $f(u) = -1$ and $f(u_1) = -2$, the edge $f^*(uu_1) < 0$. Similarly, $f(v) = 2$ and $f^*(uv) = -1$, so that $f^*(vv_1) > 0$ and $f^*(uv) > 0$. Therefore, $e_f(1) = e_f(0) + 1$. Since, positive and negative integers are equally shared among the other vertices, the other edges equally shares positive and negative labels. That is, $e_f(0) = e_f(1)$. Hence, $|e_f(0) – e_f(1)| = 1$.

CASE 2. m is odd and n is even.

We define $f : V \rightarrow \left[-\left(\frac{m+n+2}{2}\right) . . \left(\frac{m+n+2}{2}\right)\right]$ as $f(u) = -1$, $f(v) = 0$ and $f(v_1) = 1$, so that $f^*(uv) < 0$ and $f^*(vv_1) > 0$. Therefore, $e_f(0) = e_f(1)$. The other vertices $u_1, u_2, . . . , u_m, v_1, v_2, . . . , v_n$ can be labeled by the integers -2, -3, . . . , $- -\left(\frac{m+n+2}{2}\right)$, 2, 3, . . . , $\left(\frac{m+n+2}{2}\right)$ can be arranged such that the vertices receives positive and negative integers. Therefore, the edges equally shares positive and negative labels. That is, $e_f(0) = e_f(1)$. Hence, $|e_f(0) - e_f(1)| = 0$. Both cases imply $|e_f(0) - e_f(1)| \leq 1$. Hence, G is I-cordial.

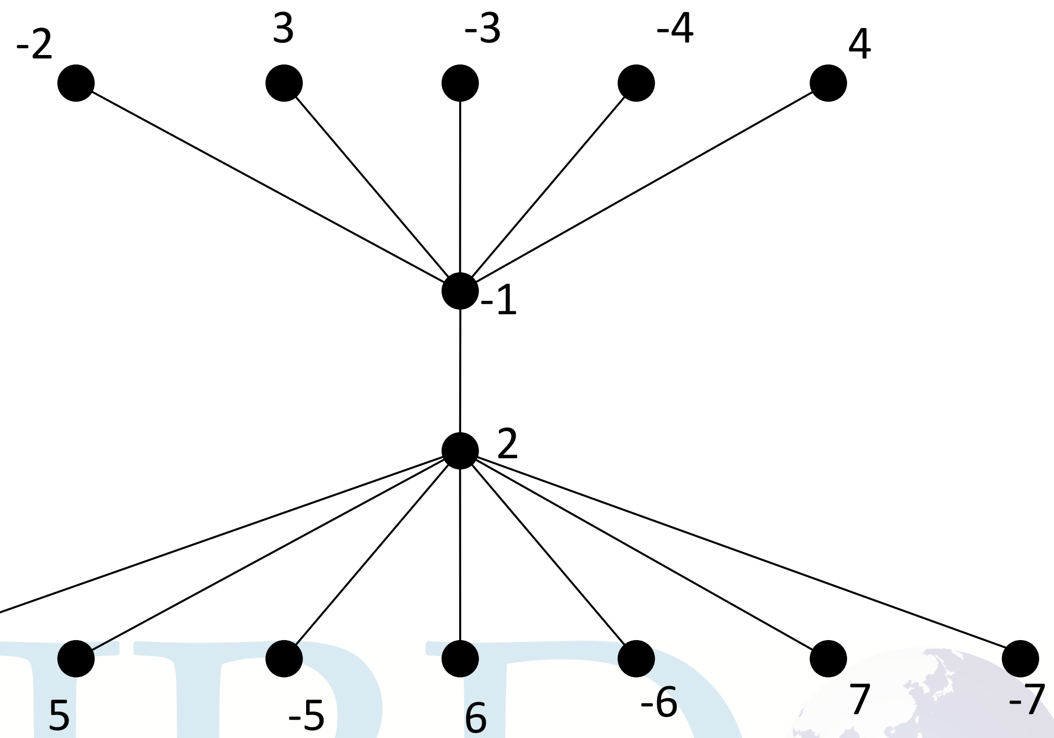

Fig. 2 $B_{5,7}$ is I-cordial.

Definition 2.5 For a graph G the ***splitting graph*** *S'(G)* of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that N(v) = N(v').

Note. The graph $S'(B_{n,n})$ is obtained from $B_{n,n}$ by adding the vertices u', v', u_i', v_i' corresponding to u, v, u_i, v_i where $1 \le i \le n$. Then $E(G) = \{uv, uv', u'v, uu_i', uu_i, u'u_i, vu_i', vv_i, v'v_i ; 1 \le i \le n\}$.

Theorem 2.6 $S'(B_{n,n})$ is an I-cordial graph.

Proof. Here p = 4(n + 1) and q = 3(2n + 1). We define $f : V \to [-2(n + 1) \, . \, . \, 2(n + 1)]^*$ as, f(u) = 2; $f(u_1') = 1$; $f(u_i') = (i + 1)$, $2 \le i \le n$; $f(u_i) = n + 1 + i$, $1 \le i \le n$; $f(u') = 2(n + 1)$; f(v) = -1; $f(v_i') = -(i + 1)$, $1 \le i \le n$; $f(v_i) = -(n + 1 + i)$, $1 \le i \le n$; $f(v') = -f(u')$.

Then $f^*(uu_i') > 0$, $f^*(uu_i) > 0$, $f^*(u'u_i) > 0$ for all i = 1, 2, . . . , n and $f^*(vv_i') < 0$, $f^*(vv_i) < 0$, $f^*(v'v_i) < 0$ for all i = 1, 2, . . . , n. Here $|e_f(0) - e_f(1)| = 0$. Since, f(u) = 2, f(v) = -1, f(u′) = 2(n + 1) and f(v′) = - 2(n+1). We have $f^*(uv) > 0$, $f^*(uv') < 0$ and $f^*(vu') > 0$. Hence $|e_f(0) - e_f(1)| = 1$.

Thus $|e_f(0) - e_f(1)| \leq 1$. Hence $S'(B_{n,n})$ is an *I*-cordial.

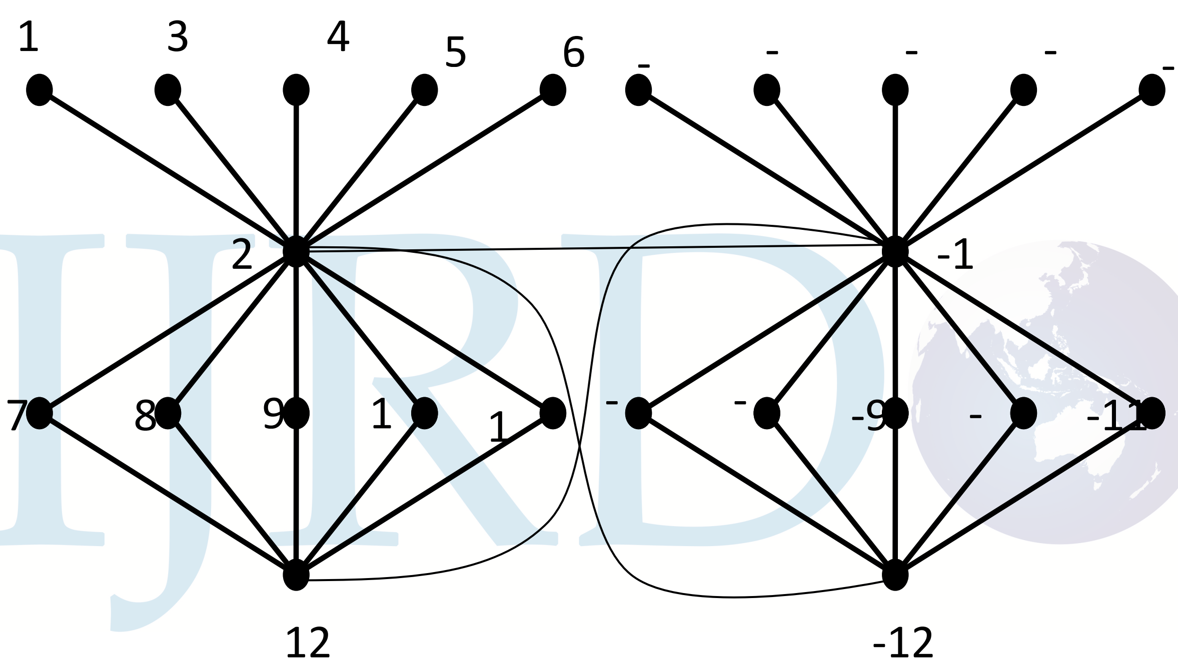

Fig. 3 $S'(B_{5,5})$ is *I*-cordial

Definition 2.7 The ***shadow graph*** $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G". Join each vertex u' in G' to the neighbors of the corresponding vertex v' in G".

Note. Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i; 1 \leq i \leq n\}$ and $\{u', v', u_i', v_i'; 1 \leq i \leq n\}$ be the corresponding vertex set of each copy of $B_{n,n}$. Let G be the graph $D_2(B_{n,n})$ and $E(G) = \{uv, uv', u'v, u'v', uu_i, u'u_i, vv_i, v'v_i ; 1 \leq i \leq n\}$then p = 4(n + 1)and q = 4(2n + 1).

Theorem 2.8 $D_2(B_{n,n})$ is I-cordial.

Proof. We define vertex labeling $f : V \rightarrow [-2(n+1) \,.\,.\, 2(n+1)]^*$ as $f(u) = 1$; $f(u') = -1$; $f(u_i) = -(i+1)$, $1 \leq i \leq n$; $f(u_i') = f(v_n) - i$, $1 \leq i \leq n$; $f(v) = 2(n+1)$; $f(v') = -2(n+1)$; $f(v_i) = -(i+1)$, $1 \leq i \leq n$; $f(v_i') = f(u_n) + i$, $1 \leq i \leq n$.

Then $f^*(uu_i) > 0$; $f^*(u'u_i') > 0$; $f^*(uui') > 0$; $f^*(u'u_i) > 0$ and $f^*(vv_i) < 0$; $f^*(v'v_i) < 0$ for all $i = 1, 2, \ldots, n$. Hence $|e_f(0) - e_f(1)| = 0$. Also, $f^*(uv) > 0$, $f^*(u'v) > 0$, $f^*(uv') < 0$, $f^*(u'v') < 0$. Hence $|e_f(0) - e_f(1)| = 0$.Thus from all the cases, $|e_f(0) - e_f(1)| \leq 1$.

Therefore, $D_2(B_{n,n})$ admits I-cordial. ■

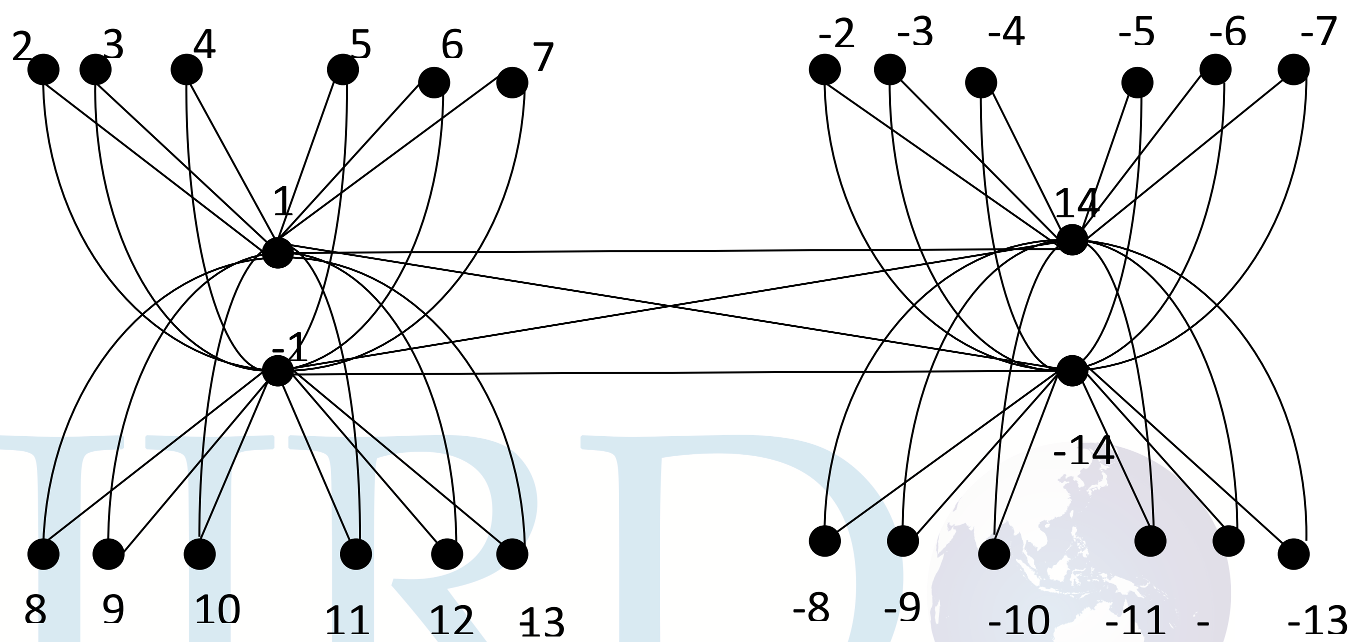

Fig. 4 $D_2(B_{6,6})$ is I-cordial

Theorem 2.9 $B^2_{n,n}$is not I-cordial graph.

Proof.Consider $B_{n,n}$ with vertex set {u, v, u_i,v_i, $1 \leq i \leq n$ } where u_i, v_i are pendant vertices. Let G be the graph $B_{n,n}{}^2$ and E(G) = {uv, vv_i, u_i, uv_i, vu_i , $1 \leq i \leq n$}, then p = 2(n+1) and q = 4n + 1. Since the vertices u and v are adjacent to u_i's and v_i's for all i = 1, 2, . . . ,n. That is, d(u) = d(v) = p -1. Then by Theorem 2.2.3, G is not I-cordial.

Theorem. 2.10 The complete bipartite graph $K_{n,n}$, $n \geq 2$ is I-cordial only if n is even.

Proof. Let $G = K_{n,n}$ be a complete bipartite graph with the partitions {U, V} where $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{v_1, v_2, \dots, v_n\}$. Then p = 2n and q = n^2. We define $f : V \rightarrow [-n..n]^*$ as follows:

$$f(u_i) = \begin{cases} -\left(\frac{i+1}{2}\right) & \text{if i is odd} ; 1 \leq i \leq n \\ \frac{i}{2} & \text{if i is even} ; 1 \leq i \leq n \end{cases}$$

$$f(v_i) = \begin{cases} -\left(\frac{n+i+1}{2}\right) & \text{if i is odd} ; 1 \leq i \leq n \\ \left(\frac{n+i}{2}\right) & \text{if i is even} ; 1 \leq i \leq n \end{cases}$$

Then $f^*(u_1v_i) > 0$ for all i = 1, 3, 5, …, n-1 and $f^*(u_1v_i) < 0$ for all i = 2, 4, 6, . . ., n. Hence n/2 edges with positive labels are incident with u_1. Similarly, n/2 edges with negative labels are incident with u_1. Similar argument holds for all n edges. That is $e_f(0) = e_f(1) = n^2/2$. Hence $|e_f(0) - e_f(0)| = 0$. Thus G is I-cordial when n is even.

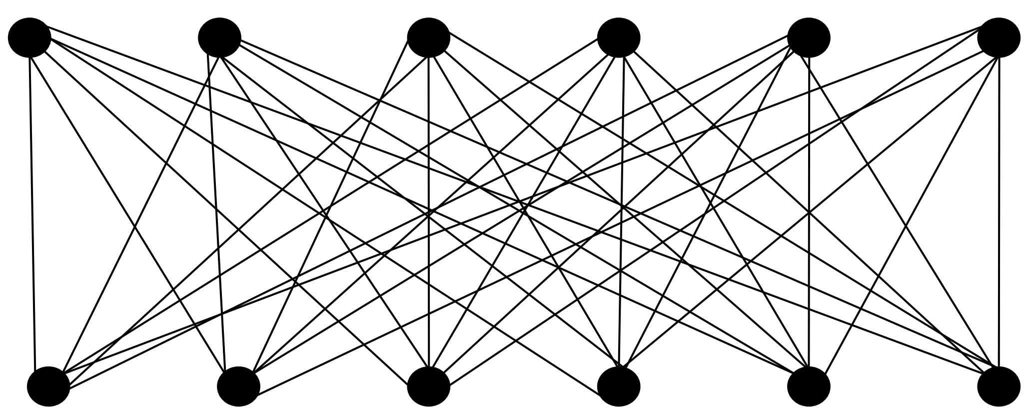

Fig. 5 $K_{6,6}$ is I-cordial

Conversly, when n is odd, the labels cannot be shared between the two partite sets in such a way that same labels of different parity lies in different partite sets. Hence $K_{n,n}$ is not an I-cordial graph when n is odd. ■

Theorem 2.11 The bipartite graph $K_{m,n}$ is I-cordial if and only if both m and n are not odd.

Proof. Consider a bipartite graph $K_{m,n}$ with $\{U, V\}$ vertices where $U = \{u_1, u_2, u_3 \ldots . u_m\}$ and $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Then p = m + n and q = mn.

CASE 1.m is even and n is odd.

We define f: $V \rightarrow \left[-\left(\frac{m+n-1}{2}\right)..\left(\frac{m+n-1}{2}\right)\right]$ as follows:

$$f(u_i) = \begin{cases} \frac{i+1}{2} & \text{if i is odd}; \ 1 \leq i \leq m \\ -\frac{i}{2} & \text{if i is even}; \ 1 \leq i \leq m \end{cases}$$

$$f(v_i) = \begin{cases} \frac{m+i+1}{2} & \text{if i is odd}; \ 1 \leq i \leq n \\ -\frac{m+i}{2} & \text{if i is even}; \ 1 \leq i \leq n \end{cases}$$

$$f(v_n) = 0$$

Then $f^*(u_1v_i) > 0$ for all i = 1,3,5, . . . ,n and $f^*(u_1v_i) < 0$ for all i = 2, 4, 6, . . . , n – 1.

Similar argument holds for all m and n vertices in $K_{m,n}$. Thus, $\frac{q}{2}$ edges equally shares label 0 and 1. That is, $e_f(0) = e_f(1) = \frac{q}{2}$. Hence $|e_f(0)- e_f(1)| = 0$.

CASE 2.m and n are even.

We define $f: V \rightarrow \left[-\left(\frac{m+n}{2}\right)..\left(\frac{m+n}{2}\right)\right]^*$, the labeling as follows:

$$f(u_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd}; \ 1 \leq i \leq m \\ -\frac{i}{2} & \text{if } i \text{ is even}; \ 1 \leq i \leq m \end{cases}$$

$$f(v_i) = \begin{cases} \frac{m+i+1}{2} & \text{if } i \text{ is odd}; \ 1 \leq i \leq n \\ -\frac{m+i}{2} & \text{if } i \text{ is even}; \ 1 \leq i \leq n \end{cases}$$

As in Case 1, $\frac{q}{2}$ edges equally share label 0 and 1. That is, $e_f(0) = e_f(1) = \frac{q}{2}$.

Hence $|e_f(0) - e_f(1)| = 0$.

Conversly, suppose both m and n are odd. Then m + n is even, say 2k. The labels cannot be shared between the two partite set in such a way that same label of different parity lies in different partite sets. Therefore $K_{m,n}$ is not I-cordial if m and n are odd. ■

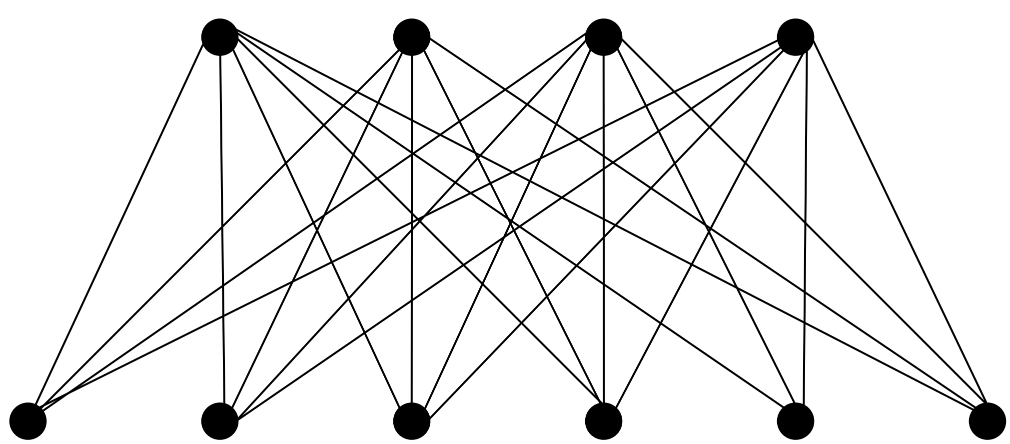

Fig. 6 $K_{4,6}$ is I-cordial.

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