ON TERNARY QUADRATIC DIOPHANTINE EQUATION

$$5(x^2+y^2)-6xy=196z^2$$

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ABSTRACT:

The ternary homogeneous quadratic equation given by $5(x^2+y^2)-6xy=196z^2$ representing a cone is analysed for its non-zero distinct integers solutions. A few interesting relations between the solutions and special polygonal numbers are presented.

KEYWORDS:

Ternary quadratic, homogeneous quadratic, integers solutions.

NOTATION USED:

$$t_{m,n}=n\left(1+\frac{(n-1)(m-2)}{2}\right)$$, A polygonal numbers of rank n with sides m.

INDRODUCTION:

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-11] for quadratic equations with three unknowns. The communication concerns with yet another interesting equation $5(x^2+y^2)-6xy=196z^2$ representing homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solution is presented.

METHOD OF ANALYSIS:

Consider the equation

$$5\left(x^2+y^2\right)-6xy=196z^2 \quad (1)$$

The substitution of linear transformations

$$x=u+v\ ;\ y=u-v \quad (u\neq v\neq 0) \quad (2)$$

in (1) leads to

$$u^2+4v^2=49z^2 \quad (3)$$

The above equation is solved through different methods and using (2), different patterns of integer solutions to (1) are obtained.

PATTERN: 1

Write 49 as

$$49=(7i)(-7i) \quad (4)$$

Assume

$$z=a^2+4b^2 \quad (5)$$

where a and b are non- zero integers

Using (4) and (5) in (3) and employing the method of factorization, define

$$u+i2v=(a+i2b)^2(7i) \quad (6)$$

Equating real and imaginary parts, we have

$$u=-28ab$$

$$2v=7a^2-28b^2$$

As our interest is on finding integer solutions, choose a and b so that u and v are integers. Replacing a by 2A and b by 2B, we have

$$z=4A^2+16B^2$$

$$u=-112AB$$

$$v=14A^2-56B^2$$

Substituting the above values of u and v in (2), the values of x and y are given by

$$\left.\begin{aligned} x=x(A,B)&=-112AB+14A^2-56B^2\\ y=y(A,B)&=-112AB-14A^2+56B^2 \end{aligned}\right\} \quad (7)$$

Thus (5) and (7) represent non-zero distinct integral solutions of (1) in two parameters.

PROPERTIES:

- $2x(A,1)+7z(A,1)-t_{114,A} \equiv 0 \pmod{169}$
- $6[2x(k^2,1)+7z(k^2,1)-t_{114,k^2}]$ is a nasty numbers
- $2x(A,1)+7z(A,1)-112t_{3,A} \equiv 0 \pmod{280}$
- $x(A,19A-17)+y(A,19A-17)+448t_{21,A}=0$
- $x(A,B)-y(A,B)+7z(A,B)-56t_{4,A}=0$

PATTERN: 2

Consider (3) as

$$u^2 = 49z^2 - 4v^2 = (7z+2v)(7z-2v) \qquad (8)$$

Write (8) in the form of ratio as

$$\frac{7z+2v}{u} = \frac{u}{7z-2v} = \frac{\alpha}{\beta}, \beta \neq 0$$

which is equivalent to the following two equations

$$\alpha u - 2\beta v - 7\beta z = 0$$

$$\beta u + 2\alpha v - 7\alpha z = 0$$

On employing the method of cross multiplication, we get

$$\frac{u}{14\alpha\beta + 14\alpha\beta} = \frac{v}{-7\beta^2 + 7\alpha^2} = \frac{z}{2\alpha^2 + 2\beta^2}$$

$$\Rightarrow \left.\begin{array}{l} u = 28\alpha\beta \\ v = 7\alpha^2 - 7\beta^2 \end{array}\right\} \qquad (9)$$

$$z = 2\alpha^2 - 2\beta^2 \qquad (10)$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y are given by

$$\left.\begin{array}{l} x = x(\alpha,\beta) = 28\alpha\beta + 7\alpha^2 - 7\beta^2 \\ y = y(\alpha,\beta) = 28\alpha\beta - 7\alpha^2 + 7\beta^2 \end{array}\right\} \qquad (11)$$

Thus (10) and (11) represent the non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

- $2x(\alpha,1)+7z(\alpha,1)-t_{58,\alpha}\equiv 0(\bmod 83)$
- $6[166x(k^2,1)+581z(k^2,1)-83t_{58,k^2}]$ is a nasty numbers
- $x(\alpha,3\alpha-1)+y(\alpha,3\alpha-1)-112t_{5,\alpha}=0$
- $x(\alpha,1)-t_{16,\alpha}\equiv 27(\bmod 34)$
- $4y(\alpha,1)-14(\alpha,1)+t_{102,\alpha}+t_{14,\alpha}\equiv 0(\bmod 58)$

PATTERN: 3

Consider (3) as

$$4v^2=49z^2-u^2=(7z+u)(7z-u) \tag{12}$$

Write (12) in the form of ratio as

$$\frac{7z+u}{4v}=\frac{v}{7z-u}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the following two equations

$$\beta u-4\alpha v+7\beta z=0$$

$$\alpha u+\beta v-7\alpha z=0$$

On employing the method of cross multiplication, we get

$$\frac{u}{28\alpha^2-7\beta^2}=\frac{v}{7\alpha\beta+7\alpha\beta}=\frac{z}{\beta^2+4\alpha^2}$$

=>

$$\left.\begin{array}{l} u=28\alpha^2-7\beta^2 \\ v=14\alpha\beta \end{array}\right\} \tag{13}$$

$$z=4\alpha^2+\beta^2 \tag{14}$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y are given by

$$\left.\begin{array}{l} x=x(\alpha,\beta)=28\alpha^2-7\beta^2+14\alpha\beta \\ y=y(\alpha,\beta)=28\alpha^2-7\beta^2-14\alpha\beta \end{array}\right\} \tag{15}$$

Thus (14) and (15) represent the non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

- $x(1,\beta)-7z(1,\beta)+t_{22,\beta}+t_{10,\beta}\equiv 0(\bmod 2)$

- $6[2x(1,k^2)-14z(1,k^2)+2t_{22,k^2}+2t_{10,k^2}]$ is a nasty numbers
- $x(2\beta-1,\beta)-y(2\beta-1,\beta)-28t_{6,\beta}=0$
- $x(1,\beta)+y(1,\beta)-14z(1,\beta)+28t_{4,\beta}=0$
- $x(5\beta-3,\beta)-y(5\beta-3,\beta)-56t_{7,\beta}=0$

PATTERN: 4

Write (12) in the form of ratio as

$$\frac{7z+u}{2v}=\frac{2v}{7z-u}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the following two equations

$$\beta u-2\alpha v+7\beta z=0$$

$$\alpha u+2\beta v-7\alpha z=0$$

On employing the method of cross multiplication, we get

$$\frac{u}{14\alpha^2-14\beta^2}=\frac{v}{7\alpha\beta+7\alpha\beta}=\frac{z}{2\beta^2+2\alpha^2}$$

=> $$\left.\begin{array}{l}u=14\alpha^2-14\beta^2\\ v=14\alpha\beta\end{array}\right\} \quad (16)$$

$$z=2\alpha^2+2\beta^2 \quad (17)$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y are given by

$$\left.\begin{array}{l}x=x(\alpha,\beta)=14\alpha^2-14\beta^2+14\alpha\beta\\ y=y(\alpha,\beta)=14\alpha^2-14\beta^2-14\alpha\beta\end{array}\right\} \quad (18)$$

Thus (17) and (18) represent the non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

- $x(1,\beta)-7z(1,\beta)+t_{42,\beta}+t_{18,\beta}\equiv 0 \pmod{12}$
- $6[-3x(1,k^2)+21z(1,k^2)-3t_{42,k^2}-3t_{18,k^2}]$ is a nasty numbers
- $x(3\beta-2,\beta)-y(3\beta-2,\beta)-28t_{8,\beta}=0$
- $2x(1,\beta)-14z(1,\beta)+t_{62,\beta}+t_{30,\beta}+t_{26,\beta}\equiv 0 \pmod{25}$
- $x(\alpha,-1)+7z(\alpha,-1)-4t_{16,\alpha}\equiv 0 \pmod{10}$

PATTERN: 5

Write (12) in the form of ratio as

$$\frac{7z+u}{v}=\frac{4v}{7z-u}=\frac{\alpha}{\beta},\beta\neq 0$$

which is equivalent to the following two equations

$\beta u-\alpha v+7\beta z=0$

$\alpha u+4\beta v-7\alpha z=0$

On employing the method of cross multiplication, we get

$$\frac{u}{7\alpha^2-28\beta^2}=\frac{v}{7\alpha\beta+7\alpha\beta}=\frac{z}{4\beta^2+\alpha^2}$$

=> $$\left.\begin{array}{l} u=7\alpha^2-28\beta^2 \\ v=14\alpha\beta \end{array}\right\} \quad (19)$$

$$z=\alpha^2+4\beta^2 \quad (20)$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y are given by

$$\left.\begin{array}{l} x=x(\alpha,\beta)=7\alpha^2-28\beta^2+14\alpha\beta \\ y=y(\alpha,\beta)=7\alpha^2-28\beta^2-14\alpha\beta \end{array}\right\} \quad (21)$$

Thus (20) and (21) represent the non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

- $x(\alpha,1)+7z(\alpha,1)-t_{30,\alpha}\equiv 0 \pmod{27}$
- $6[3x(k^2,1)+21z(k^2,1)-3t_{30,k^2}]$ is a nasty numbers
- $x(\alpha,9\alpha-7)-y(\alpha,9\alpha-7)-56t_{11,\alpha}=0$
- $2x(\alpha,1)+14z(\alpha,1)-t_{58,\alpha}\equiv 0 \pmod{55}$
- $x(\alpha,11\alpha-9)-y(\alpha,11\alpha-9)-56t_{13,\alpha}=0$

PATTERN: 6

Write (3) as

$$u^2+(2v)^2=(7z)^2 \quad (22)$$

which is in the form of well-known pythagoren equation and it is satisfied by

$$u = p^2 - q^2$$

$$v = pq$$

$$7z = p^2 + q^2$$

As our interest is on finding integer solutions, choose p and q so that u, v and z are integers. Replacing p by 7P and q by 7Q, we have

$$\left.\begin{array}{l} u = 49(P^2 - Q^2) \\ v = 49PQ \end{array}\right\} \quad (23)$$

$$z = 7(P^2 + Q^2) \quad (24)$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y given by

$$\left.\begin{array}{l} x = x(P,Q) = 49(P^2 - Q^2) + 49PQ \\ y = y(P,Q) = 49(P^2 + Q^2) - 49PQ \end{array}\right\} \quad (25)$$

Thus (24) and (25) represent the non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

- $x(1,Q) - 7z(1,Q) + t_{62,Q} + t_{52,Q} + t_{88,Q} \equiv 0 \pmod{46}$
- $6[-46x(1,k^2) + 322z(1,k^2) - 46t_{62,k^2} - 46t_{52,k^2} - 46t_{88,k^2}]$ is a nasty numbers
- $x(17Q-15,Q) - y(17Q-15,Q) - 196t_{19,Q} = 0$
- $x(P,1) - 98t_{3,P} + 49 = 0$
- $x(P,3P-1) - y(P,3P-1) - 196t_{5,P} = 0$

PATTERN: 7

Observe that (22) is also satisfied by

$$u = 2pq$$

$$2v = p^2 - q^2$$

$$7z = p^2 + q^2$$

As our interest is on finding integer solutions, choose p and q so that u, v and z are integers. Replacing p by 14P and q by 14Q, we have

$$\left.\begin{array}{l} u = 392PQ \\ v = 98(P^2 - Q^2) \end{array}\right\} \quad (26)$$

$$z = 28(P^2 + Q^2) \tag{27}$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y given by

$$\left.\begin{aligned} x = x(P,Q) = 392PQ + 98(P^2 - Q^2) \\ y = y(P,Q) = 392PQ - 98(P^2 - Q^2) \end{aligned}\right\} \tag{28}$$

Thus (27) and (28) represent the non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

- $2x(P,1) + 7z(P,1) - 784t_{3,P} \equiv 0 \pmod{392}$
- $6[4x(k^2,1) + 14z(k^2,1) - 1568t_{3,k^2}]$ is a nasty numbers
- $x(P,21P-19) + y(P,21P-19) - 1568t_{23,P} = 0$
- $x(P,9P-8) + y(P,9P-8) - 784t_{20,P} = 0$
- $x(P,1) - y(P,1) - 4(t_{22,P} + t_{21,P} + t_{12,P}) \equiv 74 \pmod{86}$

PATTERN: 8

Consider (3) as

$$u^2 + 4v^2 = 49z^2 * 1 \tag{29}$$

Write 1 as

$$1 = \frac{(3+4i)(3-4i)}{25} \tag{30}$$

Using (4), (5), (30) in (29) and employing the method of factorization, define

$$u + i2v = (7i)(a + i2b)^2 \frac{(3+4i)}{5}$$

Equating real and imaginary parts, we have

$$u = \frac{7}{5}(-4a^2 + 16b^2 - 12ab)$$

$$2v = \frac{7}{5}(3a^2 - 12b^2 - 16ab)$$

As our interest is on finding integer solutions, choose a and b so that u and v are integers. Replacing a by 10A, b by 5B, we have

$$\left.\begin{aligned} u = 7(-80A^2 + 80B^2 - 120AB) \\ v = 7(30A^2 - 30B^2 - 80AB) \end{aligned}\right\} \tag{31}$$

and also,

$$z = 100A^2 + 100B^2 \quad (32)$$

Substituting the values of u and v in (2), the non-zero distinct integral values of x and y are given by

$$\left.\begin{aligned} x = x(A,B) = -350A^2 + 350B^2 - 1400AB \\ y = y(A,B) = -770A^2 + 770B^2 - 280AB \end{aligned}\right\} \quad (33)$$

Thus (32) and (33) represent the non-zero distinct integer solutions of (1) in two parameters.

PROPERTIES:

- $2x(1,B) + 7z(1,B) - t_{2802,B} + 1401B = 0$
- $6[-2x(1,k^2) - 7z(1,k^2) + t_{2802,k^2}]$ is a nasty numbers.
- $10y(1,B) + 77z(1,B) - t_{30802,B} - 12599B = 0$

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CONCLUSION:

In this paper we have made an attempt to obtained all integer solutions satisfying the cone given by $5(x^2+y^2)-6xy=196z^2$. As Diophantine equations are infinitely many, one may search for integers solutions to other choices of quadratic and higher degree equations with many variables.