APPLICATION OF LINE GRAPHS AND COMPLETE HAMILTONIAN GRAPHS

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ABSTRACT

In 1856, Hamiltonian introduced the Hamiltonian Graph where a Graph which is covered all the vertices without repetition and end with starting vertex. In this Paper I would like to prove that Let *"G"* be a Complete graph with at least four vertices. Then, the line graph "$L(G)$" is Complete Hamiltonian if and only if *"G"* is dominating trailable.

Key Words : Graph, Hamiltonian Graph, Complete Graph, Neighborhood, Locally Complete Graph.

Introduction :

Graphs, considered here, are finite, undirected and simple and complete Graphs being followed for terminology and notation. let $G = (V , E)$ be a graph, with V the set of vertices and E the set of edges. Suppose that W is a nonempty subset of V . The sub graph of G, whose vertex set is W and whose edge set is the set of those edges of G that have both ends in W , is called the sub graph of G *induced* by W and is denoted by $G[W]$. For any vertex v in V, the *neighbour set* of v is the set of all vertices adjacent to v. This set is denoted by $N(v)$. For a graph $G = (V, E)$, we shall denote

$$\delta(G) = \min_{v \in V} |N(v)| \qquad \Delta(G) = \max_{v \in V} |N(v)|$$

a graph G = (V , E) is locally complete , if for each vertex v the graph G[N(v)] is complete . With every graph G, having at least one edge, there exists associated a graph L(G), called the line graph of G, whose vertices, can be put in a one-to-one correspondence with the edges of G, in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent.

The neighborhood is often denoted $N_G(v)$ or (when the graph is unambiguous) $N(v)$. The same neighborhood notation may also be used to refer to sets of adjacent vertices rather than the corresponding induced sub graphs. The neighborhood described above does not include *v* itself, and is more specifically the **open neighborhood** of *v*; it is also possible to define a neighborhood in which *v* itself is included, called the **closed neighborhood** and denoted by $N_G[v]$. When stated without any qualification, a neighborhood is assumed to be open.

1.1 Definition: A graph – usually denoted G(V,E) or G = (V,E) – consists of set of vertices V together with a set of edges E. The number of vertices in a graph is usually denoted *n* while the number of edges is usually denoted *m*.

1.2 Definition: Vertices are also known as nodes, points and (in social networks) as actors, agents or players.

1.3 Definition: Edges are also known as lines and (in social networks) as ties or links. An edge e = (u,v) is defined by the unordered pair of vertices that serve as its end points.

1.4 Example: The graph depicted in Figure 1 has vertex set V={a,b,c,d,e.f} and edge set

E = {(a,b),(b,c),(c,d),(c,e),(d,e),(e,f)}.

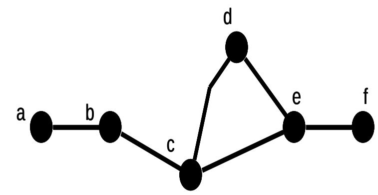

Figure 1.

1. 5 Definition: Two vertices *u* and *v* are *adjacent* if there exists an edge (*u*,*v*) that connects them.

1.6 Definition: An edge (u,v) is said to be *incident* upon nodes u and v.

1.7 Definition: An edge e = (u,u) that links a vertex to itself is known as a *self-loop* or *reflexive* tie.

1.8 Definition: Every graph has associated with it an *adjacency matrix*, which is a binary *n*×*n* matrix A in which a_{ij} = 1 and a_{ji} = 1 if vertex vi is adjacent to vertex vj, and aij = 0 and aji = 0 otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown below.

	a	b	c	d	e	f
a	0	1	0	0	0	0
b	1	0	1	0	0	0
c	0	1	0	1	1	0
d	0	0	1	0	1	0
e	0	0	1	1	0	1
f	0	0	0	0	1	0

Adjacency matrix for graph in Figure 1.

1.9 Definition: Examining either Figure 1 or given adjacency Matrix, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be *complete*.

1.10 Definition: While not every vertex in the graph in Figure 1 is adjacent, one can construct a sequence of adjacent vertices from any vertex to any other. Graphs with this property are called *connected*.

1.11 Note: Reachability. Similarly, any pair of vertices in which one vertex can reach the other via a sequence of adjacent vertices is called *reachable*. If we determine reachability for every pair of vertices, we can construct a reachability matrix R such as depicted in Figure 2. The matrix R can be thought of as the result of applying transitive closure to the adjacency matrix A.

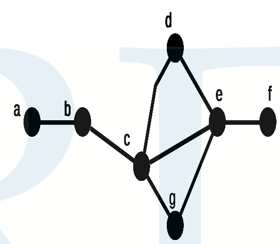

Figure: 2

1.12 Definition : A walk is closed if $v_o = v_n$.*degree* of the vertex and is denoted d(v).

1.13 Definition : A *tree* is a connected graph that contains no cycles. In a tree, every pair of points is connected by a unique path. That is, there is only one way to get from A to B.

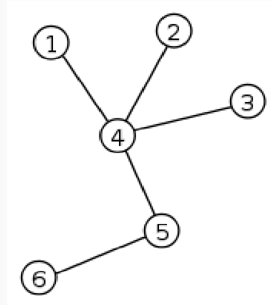

Figure 3: A labeled tree with 6 vertices and 5 edges

1.14 Definition: A *spanning tree* for a graph *G* is a sub-graph of *G* which is a tree that includes every vertex of *G*.

1.15 Definition: The length of a walk (and therefore a path or trail) is defined as the number of edges it contains. For example, in Figure 3, the path *a*,*b*,*c*,*d*,*e* has length 4.

1.16 Definition: The number of vertices adjacent to a given vertex is called the *degree* of the vertex and is denoted d(v).

1.17 Definition : In the mathematical field of graph theory, a bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

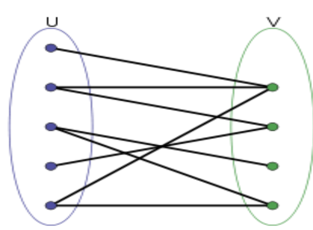

Figure 4: Example of a bipartite graph.

1.18 Definition : An Eulerian circuit in a graph G is circuit which includes every vertex and every edge of G. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once. A graph which has an Eulerian circuit is called an Eulerian graph. An Eulerian path in a graph G is a walk which passes through every vertex of G and which traverses each edge of G exactly once

1.19 Example : Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsberg exactly once?

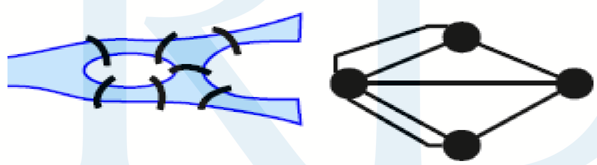

Figure 5: Konigsberg problem

2. Complete Graphs, Locally Complete Graphs, Hamiltonian Graphs, Line Graphs

In this section we have to prove that main theorem using definitions.

2.1 Definition: A Hamilton circuit is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example Hamilton Circuit would be

2.2 Example :

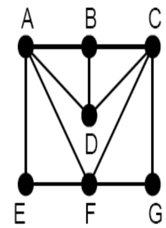

Figure: Hamilton Circuit would be AEFGCDBA.

2.3 Definition : Compete Graph: A simple graph in which there exists an edge between every pair of vertices is called a complete graph.

2.4 Definition : Let { v1, v2……vn}be the vertex set of a graph G, and for each 'α'. let Ni^{*}denote the closed neighborhood of v_a. Let N_a be any subset of N_α^{*} containing v_a which generates a complete subgraph C_a of G. Then C_a is called a complete sub neighborhood of v_a, and the indexed family $C^{*} = \{C_1, C_2, \ldots, C_n\}$ is called a complete family for G if $G = \bigcup C^{*}$. A graph G is called locally complete iff G has at least one complete family.

2.5 Examples : It is easily seen that complete graphs, trees, and unicyclic graphs are also locally complete.

The complete bigraph $K_{3,2}$ is the smallest (nontrivial, connected) graph which fails to be locally complete.

2.6 Proposition : If G is Hamiltonian, then L(G) is Hamiltonian.

Proof : This is a nice, basic result to see if a line graph is Hamiltonian.

A graph is Hamiltonian if there exists a Hamiltonian cycle in the graph.

It may be easier to find a Hamiltonian cycle in G than L(G),), but from this proposition , we would get that L(G) is Hamiltonian.

2.7 Theorem : Let 'G' be a complete graph having $n \geq 3$ vertices then L(G) is Complete Hamiltonian

2.8 Theorem : Let G be a Complete graph with at least four vertices. Then, the line graph

L(G) is Complete Hamiltonian if and only if G is dominating trail able.

Proof. We begin by assuming L(G) is Complete Hamiltonian.

So, between any two vertices, x and y, in L(G),

we have a Hamiltonian path written as

$x = x_0, x_1, x_2, \ldots, x_n = y$, where n + 1 is the number of vertices in L(G).

Since the vertices of L(G) correspond to the edges of G, then x_0, x_1, x2, ..., xn is a sequence of

edges in G, where xi$\in$ E(G) for i = 0, 1, ..., n.

Let vi be the common vertex between xi and xi+1 in G and create a list of vertices as

v0, v1, ..., vn.

Now, in this list of vertices v0, v1, ..., vn, some vertices may appear more than once.

So, create a subset w1, w2, ..., wk, where each vertex appears only once and k$\leq$ n.

In creating this, once a vertex is listed, we won't list it again.

Now, for two vertices wi and wi+1, where i = 1, 2, ..., k, list the corresponding edge between these two as ei.

Then, x1, w1, e1, w2, e2, ..., wk, ek, y is a dominating trail in G between edges

x and y,

since every edge in G is incident with one of w1, w2, ..., wk.

Since this trail works for all edges in G,

we can say that G is dominating trailable.

Conversely, we can assume G is dominating trailable, and let x and y be edges of G.

Then, there exists a dominating trail between x and y written as x, v1, x1, v2, ..., vn, xn = y, where xi $\in$ E(G) for all i = 1, 2, ..., n.

So, n is the number of internal vertices of the trail. For the remaining edges not listed in the dominating trail, we will partition in the following way.

Create n sets, labeled S1, S2, ..., Sn.

Next, for an edge incident with vi, place that edge in the corresponding set, Si.

Then, start this process with v1, and once an edge is placed in a set, do not consider that edge again. Notice that some sets may be empty, and some sets may have more than one element. Define the elements of Si as $s_{i,1}$, 1, $s_{i,2}$, 2, ..., $s_{i,r}$, r where r is the length of S_i.

Then, consider the list x, S_1, x_1, S_2, ..., Sn, y written as

x, s_1,1, 1, s_1,2, 2, ..., s_1,r, r, x_1, s_2,1, 1, ..., sn,r, r, y.

Since the edges of G correspond to

This the vertices of L(G), we now classify this sequence as a list of vertices in L(G). This sequence is a path, since it consists of distinct vertices of L(G) with each vertex in the list adjacent to the one before and after it. By construction, we have accounted

for every edge in G, and thus every vertex in L(G). makes the path

x, s_1,1, 1, s_1,2, 2, ..., s_1,r, r, x_1, s_2,1, 1, ..., sn,r, r, y a Hamiltonian path in L(G).

Since this is true for any x and y$\in$ E(G),

L(G) is Complete Hamiltonian

Hence The Theorem.

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