NONESSENTIAL PQ-INJECTIVE MODULES

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Abstract : Let M be a right $R-$module. A right $R-$module N is called *nonessential principally* M - *injective* (briefly, *nonessential* PM - *injective*) if, for each $s \in S$ with $s(M) \not\subset^e M$, any $R-$homomorphism from $s(M)$ to N can be extended to an $R-$homomorphism from M to N. M is called *nonessential principally quasi- injective* (briefly, *nonessential* PQ - *injective*) if, it is *nonessential* PM - *injective*. In this paper, we give some characterizations and properties of *nonessential* PQ - *injective* modules.

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1. Introduction

Let R be a ring. A right R -module M is called *principally injective* (or P -*injective*) [8], if every R -homomorphism from a principal right ideal of R to M can be extended to an R -homomorphism from R to M. Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$ where l and r are left and right annihilators, respectively. In [9], Nicholson, Park, and Yousif extended this notion of principally injective rings to the one for modules. In [5], W. Junchao introduced the definition of Jcp -injective rings , a ring R is called right Jcp -injective if for each $a \in R \setminus Z_r$, any R -homomorphism from aR to R can be extended to an R -homomorphism from R to R. A right R -module M is called *almost mininjective* [11] if, for any simple right ideal kR of R, there exists an S -submodule X_k of M such that $l_M(r_R(m)) = Mk \oplus X_k$ as left S -modules. A ring R is called *right almost mininjective* if R_R is almost mininjective. In this note we introduce the definition of nonessential PQ - injective modules and give some characterizations and properties. Some important results which are known for P - injective rings are hold for nonessential PQ - injective modules.

Throughout this paper, R will be an associative ring with identity and all modules are unitary right $R-$modules. For right $R-$modules M and N, $Hom_R(M, N)$ denotes the set of all $R-$homomorphisms from M to N and $S = End_R(M)$ denotes the endomorphism ring of M. If X is a subset of M the right (resp. left) annihilator of X in R (resp. S) is denoted by $r_R(X)$ (resp. $l_S(X)$). By notation, $N \subset^{\oplus} M$ $(N \subset^e M)$ we mean that N is a direct summand (an essential submodule) of M. We denote the singular submodule of M by $Z(M)$.

2. Nonessential PM - injective modules

Recall that a submodule K of a right $R-$module M is *essential* (or *large*) in M if, every nonzero submodule L of M, we have $K \cap L \neq 0$. An element $m \in M$ is called *singular* if $r_R(m) \subset^e R$. M is called *nonsingular* if it contains no nontrivial singular element.

Definition 2.1. Let M be a right $R-$module. A right $R-$module N is called *nonessential principally* M - *injective* (briefly, *nonessential* PM - *injective*) if, for each $s \in S$ with $s(M) \not\subset^e M$, any $R-$homomorphism from $s(M)$ to N can be extended to an $R-$homomorphism from M to N..

Example 2.2. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is a field, $M_R = R_R$ and $N_R = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, then N is nonessential PM - injective.

Proof. It is clear that only $X_1 = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$ and $X_3 = N$ are nonzero nonessential endomorphism images of M_R. Let $\varphi : X_1 \to N$ be an $R-$homomorphism. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in X_1$, there exists $x_{11}, x_{12} \in F$ such that $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}$.

Then $\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$.

It follows that $x_{11} = 0$.

Define $\hat{\varphi} : M \to N$ by $\hat{\varphi}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} x_{12} & 0 \\ 0 & 0 \end{pmatrix}$. It is clear that $\hat{\varphi}$ is an $R-$homomorphism.

Then $\hat{\varphi}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \hat{\varphi}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_{12} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}$.

This show that $\hat{\varphi}$ is an extension of φ. By the similar proof of X_1, we can show for X_2 and it is clear for X_3. Then N is nonessential PM - injective. □

Lemma 2.3. Let M and N be a right $R-$modules. Then N is nonessential PM - injective if and only if for each $s \in S$ with $s(M) \not\subset^e M$,

$$Hom_R(M, N)s = \{f \in Hom_R(M, N) : f(Ker(s)) = 0\}.$$

Proof. Clearly, $Hom_R(M, N)s \subset \{f \in Hom_R(M, N) : f(Ker(s)) = 0\}$.

Let $f \in Hom_R(M, N)$ such that $f(Ker(s)) = 0$. Then there exists an $R-$homomorphism $\varphi : s(M) \to N$ such that $\varphi s = f$ by Factor Theorem because $Ker(s) \subset Ker(f)$. Since N is nonessential PM - injective, there exists an $R-$homomorphism $t : M \to N$ such that $\varphi = t\iota$ where $\iota : s(M) \to M$ is the inclusion map. Hence $f = ts$ and therefore $f \in Hom_R(M, N)s$.

Conversely, let $s \in S$ with $s(M) \not\subset^e M$ and $\varphi : s(M) \to N$ be an $R-$homomorphism. Then $\varphi s \in Hom_R(M, N)$ and $\varphi s(Ker(s)) = 0$. By assumption, we have $\varphi s = us$ for some $u \in Hom_R(M, N)$. This shows that N is nonessential PM - injective. □

Lemma 2.4.

(1) If N_i $(1 \le i \le n)$ are nonessential PM - injective modules, then $\oplus_{i=1}^{n} N_i$ is nonessential PM - injective.

(2) Any direct summand of a nonessential PM - injective module is again nonessential PM - injective.

(3) If $s \in S$ with $s(M) \not\subset^e M$ and $s(M)$ is nonessential PM - injective, then $s(M) \subset^{\oplus} M$.

Proof. (1) It is enough to prove the result for $n = 2$. Let $s \in S$ with $s(M) \not\subset^e M$ and $\varphi : s(M) \to N_1 \oplus N_2$ be an $R-$homomorphism. Since N_1 and N_2 are nonessential PM - injective, there exists $R-$homomorphisms $\varphi_1 : M \to N_1$ and $\varphi_2 : M \to N_2$ such that $\varphi_1 \iota = \pi_1 \varphi$ and $\varphi_2 \iota = \pi_2 \varphi$ where π_1 and π_2 are the projection maps from $N_1 \oplus N_2$ to N_1 and N_2, respectively, and $\iota : s(M) \to M$ is the inclusion map. Put $\hat{\varphi} = \iota_1 \varphi_1 + \iota_2 \varphi_2 : M \to N_1 \oplus N_2$. Thus it is clear that $\hat{\varphi}$ extends φ.

(2) By definition.

(3) Since $s(M)$ is nonessential PM - injective, there exists an $R-$homomorphism $\varphi : M \to s(M)$ such that $\varphi \iota = 1_{s(M)}$ where $\iota : s(M) \to M$ is the inclusion map. Then by [1, Lemma 5.1], ι is a split monomorphism, therefore $s(M) \subset^{\oplus} M$. □

Theorem 2.5. The following conditions are equivalent for a projective modules M.

(1) Every $s \in S$ with $s(M) \not\subset^e M$, $s(M)$ is projective.

(2) Every factor module of a nonessential PM - injective module is nonessential PM - Injective.

(3) Every factor module of an injective $R-$module is nonessential PM - injective.

Proof. $(1) \Rightarrow (2)$ Let N be a nonessential PM - injective module, X a submodule of N, $s \in S$ with $s(M) \not\subset^e M$, and $\varphi : s(M) \to N / X$ be an $R-$homomorphism. Then by (1), there exists an $R-$homomorphism $\hat{\varphi} : s(M) \to N$ such that $\varphi = \eta \hat{\varphi}$ where $\eta : N \to N / X$ is the natural $R-$epimorphism. Since N is nonessential PM - injective, there exists an $R-$homomorphism $t : M \to N$ which is an extension of $\hat{\varphi}$ to M. Then ηt is an extension of φ to M.

$(2) \Rightarrow (3)$ is clear.

$(3) \Rightarrow (1)$ Let $s \in S$ with $s(M) \not\subset^e M$ and $\alpha : A \to B$ an $R-$epimorphism, and let $\varphi : s(M) \to B$ be an $R-$homomorphism. Embed A in an injective module E [1, 18.6]. Let $\sigma : B \to A / Ker(h)$ be an $R-$isomorphism. Since $E / Ker(\alpha)$ is nonessential PM - injective, there exists an $R-$homomorphism $\hat{\varphi} : M \to E / Ker(\alpha)$ such that $\iota_1 \sigma \varphi = \hat{\varphi} \iota_2$ where $\iota_1 : A / Ker(h) \to E / Ker(h)$ and $\iota_2 : s(M) \to M$ are the inclusion maps. Since M is projective, $\hat{\varphi}$ can be lifted to $\beta : M \to E$. Let $s(m) \in s(M)$.

Then $\sigma\varphi(s(m)) = a + \mathrm{Ker}(\alpha)$ for some $a \in A$, so $\beta(s(m)) + \mathrm{Ker}(\alpha) = \eta\beta(s(m)) = \hat{\varphi}(s(m)) = \sigma\varphi(s(m)) = a + \mathrm{Ker}(\alpha)$ where $\eta : E \to E/\mathrm{Ker}(\alpha)$ is the natural $R-$epimorphism. Hence $\beta(s(m)) - a \in \mathrm{Ker}(\alpha) \subset A$ so $\beta(s(m)) \in A$. This shows that $\beta(s(M)) \subset A$. Therefore we have lifted α. □

3. Nonessential PQ- injective modules

A right $R-$module M is called *nonessential principally quasi- injective* (briefly, *nonessential* PQ- *injective*) if, it is *nonessential* PM - *injective*.

Lemma 3.1. Let M be a right $R-$module. Then the following conditions are equivalent.

(1) M is nonessential PQ- injective.

(2) $l_S(\mathrm{Ker}(s)) = Ss$ for each $s \in S$ with $s(M) \not\subset^e M$.

(3) $\mathrm{Ker}(s) \subset \mathrm{Ker}(t)$, $s, t \in S$ and $s(M) \not\subset^e M$ implies that $St \subset Ss$.

(4) $l_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s)) = l_S(\mathrm{Im}(t) + Ss$ for $s, t \in S$ with $st(M) \not\subset^e M$.

Proof. $(1) \Rightarrow (2)$ Clearly, $Ss \subset l_S(\mathrm{Ker}(s))$ for all $s \in S$ with $s(M) \not\subset^e M$. Let $t \in l_S(\mathrm{Ker}(s))$ and define $\varphi : s(M) \to t(M)$ by $\varphi(s(m)) = t(m)$ for every $m \in M$. Then φ is well-defined because $\mathrm{Ker}(s) \subset \mathrm{Ker}(t)$. By (1), there exists an $R-$homomorphism $\hat{\varphi} : M \to M$ such that $\hat{\varphi}\iota_1 = \iota_2\varphi$ where $\iota_1 : s(M) \to M$ and $\iota_2 : t(M) \to M$ are the inclusion maps. Hence $t = \varphi s = \hat{\varphi}s \in Ss$.

$(2) \Rightarrow (3)$ If $\mathrm{Ker}(s) \subset \mathrm{Ker}(t)$, $s, t \in S$ with $s(M) \not\subset^e M$ then $l_S(\mathrm{Ker}(t)) \subset l_S(\mathrm{Ker}(s))$. Since $St \subset l_S(\mathrm{Ker}(t))$ and by (2) $l_S(\mathrm{Ker}(s)) = Ss$, so we have $St \subset Ss$.

$(3) \Rightarrow (4)$ Clearly, $l_S(\mathrm{Im}(t) + Ss \subset l_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s))$ for $s, t \in S$ with $st(M) \not\subset^e M$. Let $\varphi \in l_S(\mathrm{Im}(t) \cap \mathrm{Ker}(s))$. Then $\mathrm{Ker}(st) \subset \mathrm{Ker}(\varphi t)$, and so $S\varphi t \subset Sst$ by (3) because $st(M) \not\subset^e M$. Thus $\varphi t = \hat{\varphi}st$, $\hat{\varphi} \in S$ so $(\varphi - \hat{\varphi}s) \in l_S(\mathrm{Im}(t))$. It follows that $\varphi \in l_S(\mathrm{Im}(t) + Ss$.

$(4) \Rightarrow (1)$ Let $s \in S$ with $s(M) \not\subset^e M$ and $\varphi : s(M) \to M$ be an $R-$homomorphism. Then $\varphi s \in l_S(\mathrm{Ker}(\varphi s)) \subset l_S(\mathrm{Ker}(s)) = l_S(\mathrm{Ker}(s) \cap \mathrm{Im}1) = l_S(\mathrm{Im}1) + Ss = Ss$ by (4) because $s1(M) \not\subset^e M$. Thus there exists an $R-$homomorphism $\hat{\varphi} \in S$ is an extension of φ to M. □

Following [8], a right $R-$module M is called a *duo module* if every submodule of M is fully invariant.

Theorem 3.2. Let M be a duo, nonessential PQ- injective module and $s, t \in S$ with $s(M) \not\subset^e M$.

(1) If $s(M)$ embeds into $t(M)$, then Ss is an image of St.

(2) If $t(M)$ is an image of $s(M)$, then St can be embedded into Ss.

(3) If $s(M) \simeq t(M)$, then $Ss \simeq St$.

Proof. (1) Let $f:s(M)\to t(M)$ be an $R-$monomorphism. Since M is nonessential PQ- injective, there exists an $R-$homomorphism $\hat{f}:M\to M$ such that $\hat{f}\iota_1=\iota_2 f$ where $\iota_1:s(M)\to M$ and $\iota_2:t(M)\to M$ are the inclusion maps. Let $\sigma:St\to Ss$ defined by $\sigma(ut)=u\hat{f}s$ for every $u\in S$. Since $\hat{f}s(M)\subset t(M)$, σ is well-defined. It is clear that σ is an $S-$homomorphism. Since $\hat{f}\big|_{s(M)}$ is monic and M is a duo module, $\hat{f}(s(M))\subset s(M)$ so $fs(M)\not\subset^e M$. Since $Ker(fs)\subset Ker(s)$, $Ss\subset Sfs$ by Lemma 3.1. Then $s\in Sfs\subset\sigma(St)$.

(2) By the same notations as in (1), let $f:s(M)\to t(M)$ be an $R-$epimorphism. Since M is nonessential PQ- injective, there exists an $R-$homomorphism $\hat{f}:M\to M$ such that $\hat{f}\iota_1=\iota_2 f$. Let $\sigma:St\to Ss$ defined by $\sigma(ut)=u\hat{f}s$ for every $u\in S$. It is clear that σ is an $S-$homomorphism. If $ut\in Ker(\sigma)$, then $0=\sigma(ut)=u\hat{f}s=ufs$. It follows that $ut=0$.

(3) Follows from (1) and (2) □

Recall that a right R-module M is called C2 [6] if, every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. M is called C3 if whenever N and K are direct summands of M with $N\cap K=0$ then $N\oplus K$ also a direct summand of M.

Theorem 3.3. Let $M=mR$, $m\in M$ be a principal, nonessential PQ- injective module.

(1) If $nR\simeq e(mR)$ where $n\in M$ and $1\neq e=e^2\in S$, then $nR=g(mR)$, for some $g=g^2\in S$.

(2) If $e(mR)\cap f(mR)=0$, $1\neq e=e^2\in S$, $1\neq f=f^2\in S$, then $e(mR)\oplus f(mR)=g(mR)$, For some $g=g^2\in S$.

Proof. (1) If $nR\simeq e(mR)$ where $n\in M$ and $1\neq e=e^2\in S$, then $e(mR)$ is nonessential PM- injective by Lemma 2.4 and hence nR is also nonessential PM- injective. Since $nR\simeq e(mR)$, there exists an isomorphism σ such that $nR\simeq\sigma e(mR)$. Since $\sigma e(M)\not\subset^e M$, then $nR\subset^{\oplus} M$ Lemma 2.4.

(2) Let $e(mR)\cap f(mR)=0$, $1\neq e=e^2\in S$, $1\neq f=f^2\in S$. Then $e(M)\oplus f(M)=e(M)\oplus(1-e)f(M)$. Since $(1-e)f(M)\simeq f(M)$, $(1-e)f(M)=g(M)$ for some $g^2=g\in S$ by (1). Let $h=e+g-ge$, then $h^2=h$ and $e(M)\oplus f(M)=h(M)$. This prove (2). □

Theorem 3.4. Let M be a principal, nonessential PQ- injective, quasi-projective module and $s\in S$ with $s(M)\not\subset^e M$. Then the following conditions are equivalent.

(1) $s(M)$ is a direct summand of M.

(2) $s(M)$ is $M-$projective.

(3) $s(M)$ is nonessential PQ- injective.

Proof. (1) $\Rightarrow$ (2) It follows from the projectivity of M.

$(2)\Rightarrow(3)$ Since the sequence $0\to \mathrm{Ker}(s)\to M\to s(M)\to 0$ splits, $s(M)$ is isomorphic to a direct summand of M so it is nonessential PM - injective by Theorem 3.3 and Lemma 2.4.

$(3)\Rightarrow(1)$ It follows from Lemma 2.4. □

Definition 3.5. Let M be a right R-module, $S=\mathrm{End}_R(M)$. The module M is called *almost nonessential* PQ- *injective* if, for each $s\in S$ with $s(M)\not\subset^e M,$ there exists a left ideal X_s of S such that $l_S(r_M(s)) = Ss\oplus X_s$ as left S-modules.

Lemma 3.6. Let M be a right R-module, $S=\mathrm{End}_R(M)$ and $s\in S$ with $s(M)\not\subset^e M.$

(1) If $\mathrm{Hom}_R(s(M),M) = S\oplus Y$ as left S-modules, then $l_S(\mathrm{Ker}(s)) = Ss\oplus X$ as left S- modules, where $X=\{fs: f\in Y\}.$

(2) If $l_S(\mathrm{Ker}(s)) = Ss\oplus X$ for some $X\subset S$ as left S- modules, then we have $\mathrm{Hom}_R(s(M),M) = S\oplus Y$ as left S- modules, where $Y=\{f\in \mathrm{Hom}_R(s(M),M): fs\in X\}.$

(3) Ss is a direct summand of $l_S(\mathrm{Ker}(s))$ as left S- modules if and only if S is a direct summand of $\mathrm{Hom}_R(s(M),M)$ as left S- modules.

Proof. Define $\theta:\mathrm{Hom}_R(s(M),M)\to l_S(\mathrm{Ker}(s))$ by $\theta(f)=fs$ for every $f\in \mathrm{Hom}_R(s(M),M)$ It is obvious that θ is an S-monomorphism. For $t\in l_S(\mathrm{Ker}(s))$ define $g: s(M)\to M$ by $g(s(m))=t(m)$ for every $m\in M.$ Since $\mathrm{Ker}(s)\subset \mathrm{Ker}(t),$ g is well-defined, so it is clear that g is an R-homomorphism. Then $\theta(g)=gs=t.$ Therefore θ is an S-isomorphism. Let $fs\in Ss.$ Since $fs\in l_S(\mathrm{Ker}(s)),$ there exists $\varphi\in \mathrm{Hom}_R(s(M),M)$ such that $\theta(\varphi)=fs,$ so $\varphi s=fs.$ Define $\hat{\varphi}:M\to M$ by $\hat{\varphi}(m)=f(m)$ for every $m\in M.$ It is clear that $\hat{\varphi}$ is an R-homomorphism and is an extension of $\varphi.$ Then $fs=\hat{\varphi}s=\theta(\hat{\varphi}).$ This shows that $Ss\subset\theta(S).$ The other inclusion is clear. Then $\theta(S)=Ss$ and $X=\theta(Y)=\{fs: f\in Y\}.$ Then the Lemma follows. □

Theorem 3.7. The following conditions are equivalent:

(1) M is almost nonessential PQ- injective.

(2) There exists an indexed set $\{X_s : s\in S\}$ of left ideals of S with the property that if $s(M)\not\subset^e M,$ $s\in S,$ then $l_S(\mathrm{Im}(t)\cap \mathrm{Ker}(s)) = (X_{st}:t)_l + Ss$ and $(X_{st}:t)_l\cap Ss\subset l_S(t)$ for all $t\in S,$ where $(X_{st}:t)_l=\{g\in S: gt\in X_{st}\}$ if $st\neq 0$ and $(X_{st}:t)_l=l_S(\mathrm{Im}(t))$ if $st=0.$

Proof. $(1)\Rightarrow(2)$ Let $s\in S$ with $s(M)\not\subset^e M.$ Then there exists a left ideal X_s of S such that $l_S(\mathrm{Ker}(s)) = Ss\oplus X_s$ as left S-modules. Let $t\in S.$ If $st=0,$ then $\mathrm{Im}(t)\subset \mathrm{Ker}(s)$ so (2) follows. If $st\neq 0,$ then any $g\in l_S(\mathrm{Im}(t)\cap \mathrm{Ker}(s))$ we have $\mathrm{Ker}(st)\subset \mathrm{Ker}(gt)$ and so $gt\in l_S(\mathrm{Ker}(gt))\subset l_S(\mathrm{Ker}(st)) = Sst\oplus X_{st}$ as left S-modules because $st(M)\not\subset^e M.$ Write $gt=\alpha(st)+h$ where $\alpha\in S$ and $h\in X_{st}.$ Then

$(g-\alpha(s))t = h \in X_{st}$, so $g-\alpha s \in (X_{st}:t)_l$ It follows that $g \in (X_{st}:t)_l + Ss$. This shows that $l_S(Ker(s)\cap Im(t)) \subset (X_{st}:t)_l + Ss$. Conversely, it is clear that $Ss \subset l_S(Ker(s)\cap Im(t))$. Let $h \in (X_{st}:t)_l$. Then $ht \in X_{st} \subset l_S(Ker(st))$. If $t(m) \in Ker(s)\cap Im(t)$, then $st(m)=0$ and so $ht(m)=0$. Hence $h \in l_S(Ker(s)\cap Im(t))$. This shows that $(X_{st}:t)_l \subset l_S(Ker(st))$. Therefore $l_s(Ker(s)\cap Im(t)) = (X_{st}:t)_l + Ss$. If $\beta s \in (X_{st}:t)_l \cap Ss$, then $\beta st \in X_{st} \cap Sst = 0$. Hence $\beta s \in l_S(t)$.

$(2) \Rightarrow (1)$ Let $s \in S$ with $s(M) \not\subset^e M$. Then there exists a left ideal X_S of S such that $l_S(Ker(s)) = l_S(Ker(s)\cap Im(1)) = (X_s:1)_l + Ss$ and $(X_s:1)_l \cap Ss \subset l_S(1) = 0$.
Note that $(X_s:1)_l = X_s$. Then (1) follows. □

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