Satellite-only Versus Combined High Degree Global Geopotential Model Tailored to Gravity Field in Egypt, Using Integral Formulas

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Abstract

Global Geopotential Models (GGMs) may contain long-wavelength errors due to difficulties in collecting and using global heterogeneous gravity data, which further degrades the quality of regional gravity field modelling.

In this study, satellite-only and high degree reference geopotential models denoted as GOCO05s and EGM2008, respectively, have been tailored (refined) to fit the gravity data in Egypt using integral formulas in order to select the optimal model that can be used for the reference gravity field model for the Egyptian territory.

The results show that the tailored model of EGM2008 denoted as EGTM0817 is the one that gives better results in Egypt than the other tailored model of GOCO05s denoted as EGTGOC5s, where the mean value, the standard deviation and the range of the reduced gravity anomalies to EGTM0817 compared with EGTGOC5s have lesser values by about 80%, 30%, and 21%, respectively.

Keywords: GGMs, Tailored model, Integral technique, Covariance function.

1. Introduction

Current Global Geopotential Models (GGMs) are not perfect due to imperfect distribution, density, and accuracy of the available global heterogeneous gravity data, whereas data availability and data accuracy can only be enhanced by performing additional observations, accordingly the resolution of the geopotential models can then be improved by increasing its maximum degree. Practical studies had proved that the methods of tailoring a model (modified the model to fit regional/local gravity data) have succeeded to upgrade it as a reference model for better regional gravity field solutions see e.g. (Bašić, T. et al., 1990;Kearsley & Forsberg, 1990; Wenzel, 1998; Amin, M. et al., 2003; Abd-Elmotaal, 2014).

Hence, the aim of this research is to tailor (fit) the geopotential model to Egypt for better modelling of the Egyptian gravity field. This can be made by computing the differences between local gravity anomalies and those derived from the geopotential model, then the harmonic analysis of the residual gravity anomalies yields correction terms that are added to the original spherical coefficients of the relevant model to give the final modified coefficients of the fitted model.

Several methods can be used to achieve the tailoring process, in this paper; we have used the integral formulas, suggested by Weber and Zomorrodian (1988), to tailor the satellite-only model GOCO05s (Mayer et al., 2015) and ultra-high degree geopotential model EGM2008 (Pavlis et al., 2012) for Egypt in order to determine the best of them that would be considered as reference model for better modelling of the Egyptian gravity field.

The Egyptian 5′×5′ mean free-air gravity anomalies interpolated by Least Squares Collocation (LSC) are used to estimate the harmonic coefficients of the tailored model GOCO05s denoted as EGTGOC5s complete to degree and order 280 as well as the ultra-high degree reference model EGM2008 was tailored to respective maximum degree and order 560 and has been restored the higher harmonic degrees (from $n = 561$ to $n = 2190$) for increasing the resolution of the tailored model, yielding the final tailored model for Egypt, denoted as EGTM0817.

The results show that the standard deviation of the reduced gravity anomalies to EGTGOC5s have been decreased by about 17%, compared to the corresponding quantities of the reduced data relevant to GOCO05s. As for the EGM2008 model, the standard deviation of the reduced gravity anomalies to its final tailored model EGTM0817 have been dropped by about 38%, compared with its original version. Thus, the results show that the both tailored geopotential models, give less, and better residual anomalies. This reflects that the homogenization of the tailored models on a local gravity data in Egypt.

The comparison of both tailored models EGTM0817 and EGTGOC5s, with the local point gravity anomaly, reveals that the model EGTM0817 has a better performance, in terms of the standard deviation of its residual anomalies, which is lesser by about 30.0 % than the EGTGOC5s model.

Finally, the results of this study indicate that the tailored geopotential models to the local gravity field in Egypt should have an important impact in geoid solutions. It also shows that the tailored model EGTM0817 is the best to be used for gravity field modeling in this region till now.

2. Data set

2.1. Gravity Anomalies Data

A set of 6311 points gravity anomalies is available (5739 in Land and 572 at Marine) for Egyptian Territory, was obtained from various local and international organizations such as Ganoub El- Wadi Petroleum Holding Company (Ganope), National Research Institute of Astronomy and Geophysics (NRIAG), Survey Research Institutes (SRI), General Petroleum Company (GPC), Egyptian Survey Authority (ESA) and Bureau Gravimétrique International (BGI).

This data set refers to the World Geodetic System 1984 (WGS 84) and International Gravity Standardization Net 1971(IGSN-71) gravity datum and covers the window $21^{o} \leq \varphi \leq 33^{o}$ and $24^{o} \leq \lambda \leq 38^{o}$ as shown in Fig, (1).

The distributions of point gravity anomalies are very poor on land and the distribution of the data set at the Red Sea is better than that at the Mediterranean Sea. For this reason, a major effort has been made to improve the resolution of the local Egyptian free-air anomalies by using a $30' \times 30'$ arc-minute mean value of gravity anomalies (493 terrestrial and 123 marines) made by National Imagery and Mapping Agency (NIMA) see in Fig, (1). The NIMA 30′ mean anomalies available via anonymous FTP to ftp://cddis.gsfc.nasa.gov/ (cd to the directory pub/egm96/gravity_data/).

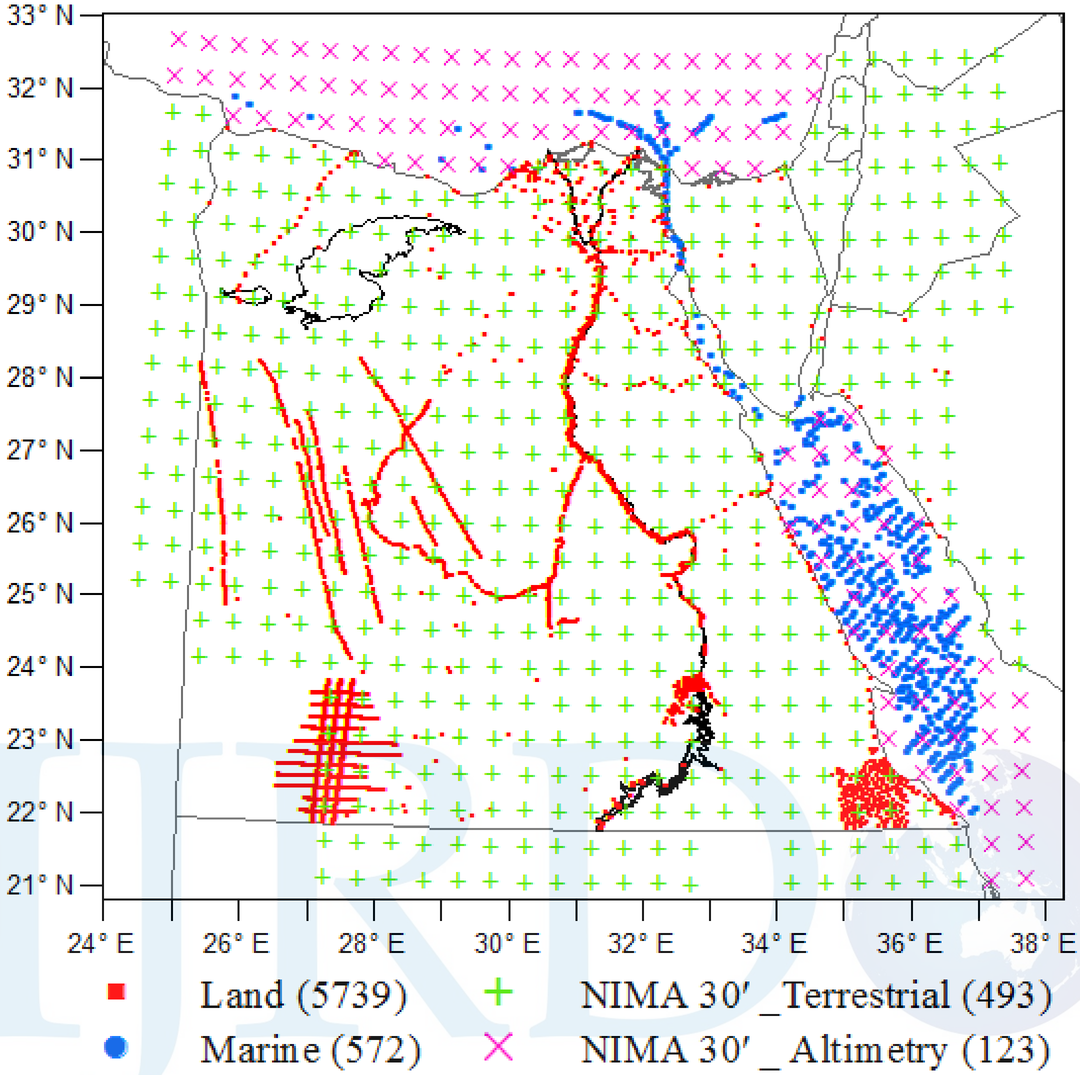

Figure 1: Distribution of the local Egyptian free-air gravity anomalies.

2.2. Global Geopotential Model:

In this study, we selected two of geopotential model for picking the one that suits the Egyptian territory better, the satellite-only model GOCO05s and the ultra-high degree reference model EGM2008. The first is selected because it signifies unsurpassed satellite-only models, which is based on complete data of the three gravity field mapping missions (CHAMP, GRACE, and GOCE), resolved up to degree/order 280 of a harmonic series expansion by the Gravity Observation Combination Consortium (GOCO), while the second is picked because it represents one of the best ultra-high degree model, released by the US National Geospatial-Intelligence Agency (NGA), usually used as a reference model to assess other the latest development geopotential models. In addition, it's complete to spherical harmonic degree and order 2159, but contains additional spherical harmonic coefficients to degree 2190 and order 2159.

3. Tailoring a Global Geopotential Model to certain area.

Improving or refining a geopotential model to fit the gravity field of the certain region using additional gravity data relevant to that area, are often referred to as tailoring the same model to this region. The basic assumption is that the additional gravity data have not been used originally in the development of the geopotential model. To achieve this process, the differences between the additional gravity data and those obtained from the geopotential model of interest are used in harmonic analysis techniques to obtain correction terms that are added to the coefficients of the original model to give the final refined coefficients of the tailored model as follows:

$$\begin{Bmatrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{Bmatrix}_{Tailored\ Model} = \begin{Bmatrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{Bmatrix}_{Original\ Model} + \begin{Bmatrix} \delta\bar{C}_{nm} \\ \delta\bar{S}_{nm} \end{Bmatrix}_{Corrections} \quad (1)$$

The computation of the tailored model Eq. (1) is carried out by integral formulas through the iterative algorithm, as will be discussed later.

3.1. Harmonic Analysis of a Global Field Using Integral Techniques.

The gravity anomaly (Δg) in spherical approximation is given (Torge, 1989, p. 44) as follows:

$$\Delta g(r,\theta,\lambda) = \frac{GM}{r^2} \sum_{n=2}^{n_{max}} (n-1) \left[\frac{a}{r}\right]^n \sum_{m=0}^{n} (\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\, \overline{P_{nm}}\,(\cos\theta) \quad (2)$$

Where r is geocentric radius, θ is polar distance, λ geodetic longitude, GM is the geocentric gravitational constant, a stands for the equatorial radius of the geocentric reference ellipsoid, $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are the fully normalized geopotential coefficients of degree n and order m, n_{max} is maximum degree, and $\bar{P}_{nm}(cos\theta)$ denotes the associated fully normalized Legendre functions. The quadrature procedure for estimating spherical harmonic coefficients may be computed from gravity anomalies Eq. (2) by employing the orthogonality relationships for fully normalized spherical harmonic functions as (ibid., 1989, p.44):

$$\begin{Bmatrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{Bmatrix} = \frac{1}{4\pi} \iint_{\sigma} \frac{r^2}{GM} \left(\frac{r}{a}\right)^n \frac{1}{n-1}\, \Delta g(r,\theta,\lambda) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \overline{P}_{nm}(\cos\theta)\, d\sigma \quad (3)$$

Where σ is a unit sphere and $d\sigma$ is surface area element. The actual method of evaluation of Eq. (3) is carried out using a set of mean gravity anomalies all over the earth's surface, however, when the integration is carried out over a local area only, it is thus implicitly assumed that the mean gravity anomalies are equal to zero outside σ.

A mean gravity anomaly can be computed from geopotential models, (Rapp, 1977, p.4) as follows:

$$\overline{\Delta g}_{Model} = \frac{GM}{r^2} \sum_{n=2}^{n_{max}} (n-1) \left[\frac{a}{r}\right]^n \beta_n \sum_{m=0}^{n} (\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda)\, \overline{P}_{nm}(\cos\theta) \qquad (4)$$

Where β_n are the Pellinen smoothing functions that can be evaluated from a recurrence relation formula derived in Sjöberg (1980).

Comparing the mean gravity anomalies derived from the geopotential model Eq. (4) with the local mean gravity anomalies, denoted by ($\overline{\Delta g}$), derived from local gravity data, yields residual gravity anomalies as,

$$\delta\overline{\Delta g} = \overline{\Delta g} - \overline{\Delta g}_{Model} \qquad (5)$$

The residual gravity anomalies $\delta\overline{\Delta g}$ can be expanded in spherical harmonic in order to yield the corrections $\delta\overline{C}_{nm}$, $\delta\overline{S}_{nm}$ by replacing it by the gravity anomalies in Eq. (3) as (Weber & Zommorrodian, 1988),

$$\begin{Bmatrix} \delta\overline{C}_{nm} \\ \delta\overline{S}_{nm} \end{Bmatrix} = \frac{1}{4\pi} \sum_{i=1}^{K} \frac{r_i^2}{GM} \left(\frac{r_i}{a}\right)^n \frac{1}{n-1} \frac{1}{\beta_n}\, \delta(\overline{\Delta g_i}) \iint_{\Delta\sigma_i} \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \overline{P}_{nm}(\cos\theta)\, d\sigma \qquad (6)$$

Where *k* is the number of iteration (between model and local anomalies).

Finally, the coefficients of the tailored model are now obtained by adding the corrections Eq. (6) to original coefficients of Eq. (1), then the residual gravity anomalies Eq. (5) may once again be formed iteratively (Kearsley & Forsberg, 1990) as follows:

$$\delta(\overline{\Delta g_i}) = \overline{\Delta g} - (\overline{\Delta g_i})_{Model} \qquad i = 1, 2, \ldots\ldots k \qquad (7)$$

Where the index *i signals the* i^{th} tailoring step. Formula (1) and (4) to (7) can be iterated until $(\delta\overline{\Delta g})_i$ no longer shows a significant decrease in its value and in its Root Mean Square (RMS) variation.

4. Computations and results

4.1. Preparing the Gravity Anomalies for tailoring

Since the approach described in the previous section requires that the gravity anomalies should be computed on a regular grid (mean value) for harmonic analysis computation, the NIMA 30′×30′ mean gravity anomalies were merged with the available point gravity anomalies to interpolate the Egyptian 5′×5′ arc-minute mean free-air gravity anomalies $(\overline{\Delta g})$ through a fast quadrant-search Least Squares Collocation (LSC) prediction algorithm, by the program GEOGRID, see Fig. (2).This is also made to achieve the maximum degree of the tailored models.

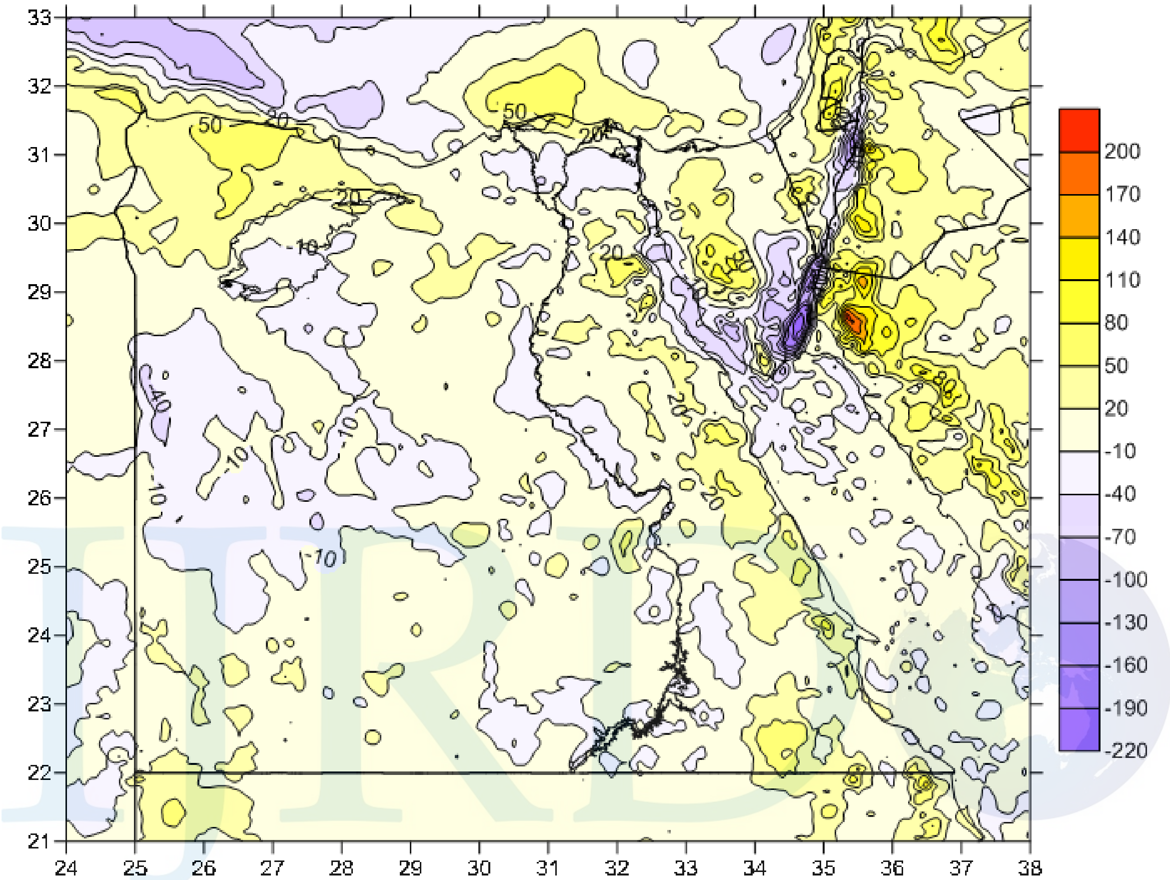

Figure 2: The Egyptian 5′×5′ mean free-air gravity anomalies interpolated by Least Squares Collocation. Contour interval 30 mGal

4.2. Computations of Tailored models

The Egyptian 5′×5′ mean free-air gravity anomalies are used to estimate the new harmonic coefficients of tailored satellite-only model GOCO05s denoted as EGTGOC5s till degree and order 280 as given in Table (1). Also, using the same 5 arc-minute anomalies, the high degree reference model EGM2008 was tailored to maximum degree 560 as given in Table (2), yielding the model denoted as EGM08T. The harmonic coefficients of tailored models have been obtaining by integral formulas through five iterations, to improve the accuracy of the obtained harmonic coefficients and to decrease the residual field, as described in section (3), by using the program PMITES after simple modifications to increase the maximum degree and order. A Comparison after each iteration has been carried out between the local mean anomalies and the computed anomalies from the tailored model until the residual field has reached its minimal value.

Table 1: Statistics of the differences of the Egyptian 5′×5′ mean gravity anomalies with GOCO05s and tailored model EGTGOC5s. (Max. degree. 280)

Gravity Anomalies	No. of iterations	Mean	RMS	Minimum	Maximum
		mGal	mGal	mGal	mGal
Mean Free-air ($\overline{\Delta g}$)		4.814	27.560	-216.060	211.341
$\overline{\Delta g}$ - GOCO05s	0	-0.726	20.382	-213.835	181.297
$\overline{\Delta g}$ - EGTGOC5s	5	0.002	13.489	-136.754	150.112

Table 2: Statistics of the differences of the Egyptian 5 arc-minute mean gravity anomalies with EGM2008 and tailored model EGM08T. (Max. degree. 560)

Gravity Anomalies	No. of iterations	Mean	RMS	Minimum	Maximum
		mGal	mGal	mGal	mGal
Mean Free-air ($\overline{\Delta g}$)		4.814	27.560	-216.060	211.341
$\overline{\Delta g}$ - EGM2008	0	-0.696	14.966	-129.506	119.277
$\overline{\Delta g}$ - EGM08T	5	0.001	8.974	-102.731	114.040

The results listed in Tables (1) and (2), show that the tailored model EGTGOC5s has improved by about 34.0 % compared to GOCO05s, while the tailored model EGM08T has significant improvement by about 40.0 % w.r.t the start model EGM2008, in terms of RMS.

For increasing the resolution of the best-tailored model EGM08T, the higher harmonic degrees (from n = 561 to n = 2190) of the original model EGM2008 have been restored, yielding the final tailored model for Egypt, denoted as EGTM0817. Once again, Tables (3) shows a comparison between the differences of the local 5′×5′ mean gravity anomalies using both models EGM2008 and EGTM0817.

Table 3: Statistics of the differences of the Egyptian 5′×5′ mean gravity anomalies with EGM2008 and tailored model EGTM0817

Mean Free-air anomalies ($\overline{\Delta g}$)		Max. Degree.	Mean	RMS	Minimum	Maximum
NO. of values	24505		mGal	mGal	mGal	mGal
$\overline{\Delta g}$ - EGM2008		2190	-0.600	12.608	-98.584	51.834
$\overline{\Delta g}$ - EGTM0817		2190	0.097	4.643	-40.883	42.393

From Tables (2) and (3), the comparison shows that the RMS of the residuals of tailored model drops from ± 8.974 mGal for EGM08T to ± 4.643 mGal for EGTM0817 i.e. about 48.0 % decrease. We can also notice that EGTM0817 has been enhanced by about 63 % compared to the original EGM2008, in terms of RMS. The last result reflects the importance of restoring the higher degrees (from n = 561 to n = 2190) that have increased the accuracy of the final tailored model EGTM0817.

Finally, the comparison between both tailored models reveals a highly better performance for the EGTM0817 over EGTGOC5s model, where the RMS of the residual mean gravity anomalies has been dropped from ± 13.489 mGal for EGTGOC5s to ± 4.643 for EGTM0817 mGal by about 66.0 %. Thus, it is clear that the final tailored model EGTM0817 fits better the local mean gravity anomalies in Egypt than EGTGOC5s.

5. Impact of the incorporation of the local gravity data on the tailored models

In order to confirm the results reached in the previous section and to detect the impact of the incorporation of the local Egyptian gravity anomalies into the original model to produce the tailored model the empirical covariance function of the residual gravity anomalies, has been made for these purposes in the following section.

5.1. Empirical Covariance functions of residual field

The available point gravity anomalies in Egypt, described in section (2), are reduced to the harmonic models GOCO05s, EGTGOC5s, EGM2008, and EGTM0817 as given in Table (5). The computations were carried out using program GEOCOL.

Table 5: Statistics of residual anomalies (6311 gravity stations, Max. degree. in brackets)

Models	Mean mGal	Standard dev. mGal	Minimum mGal	Maximum mGal	Range mGal
GOCO05s (280)	-1.233	11.131	-77.885	66.881	144.766
EGTGOC5s (280)	-1.178	9.200	-46.255	48.482	94.737
EGM2008 (2190)	-0.606	10.308	-45.365	43.776	89.031
EGTM0817 (2190)	0.229	6.398	-38.835	35.998	74.833

From Table (5) both tailored models EGTGOC5s and EGTM0817, give less, and better residual anomalies, reflecting the homogenization of these models on a local gravity data in Egypt. As a result it is obvious that the standard deviation and the range of the reduced gravity anomalies to EGTGOC5s compared with GOCO05s have been decreased by about 17% and 35%, respectively, while the standard deviation and the range of the reduced gravity anomalies to EGTM0817 compared with EGM2008 have been dropped by about 38% and 16%, respectively. The comparison between both tailored models reveals a better accuracy for the EGTM0817 model, where the mean value, the standard deviation and the range of the reduced gravity anomalies to EGTM0817 have decreased by about 80%, 30%, and 21%, respectively, compared with EGTGOC5s.

The results of above residual gravity field are then discussed regarding the corresponding empirical covariance functions, which is a function of the separation between the data points that describe the spatial variability of the local residual field. The programs EMPCOV was used for the estimation of this function, and are plotted in Fig. (5).

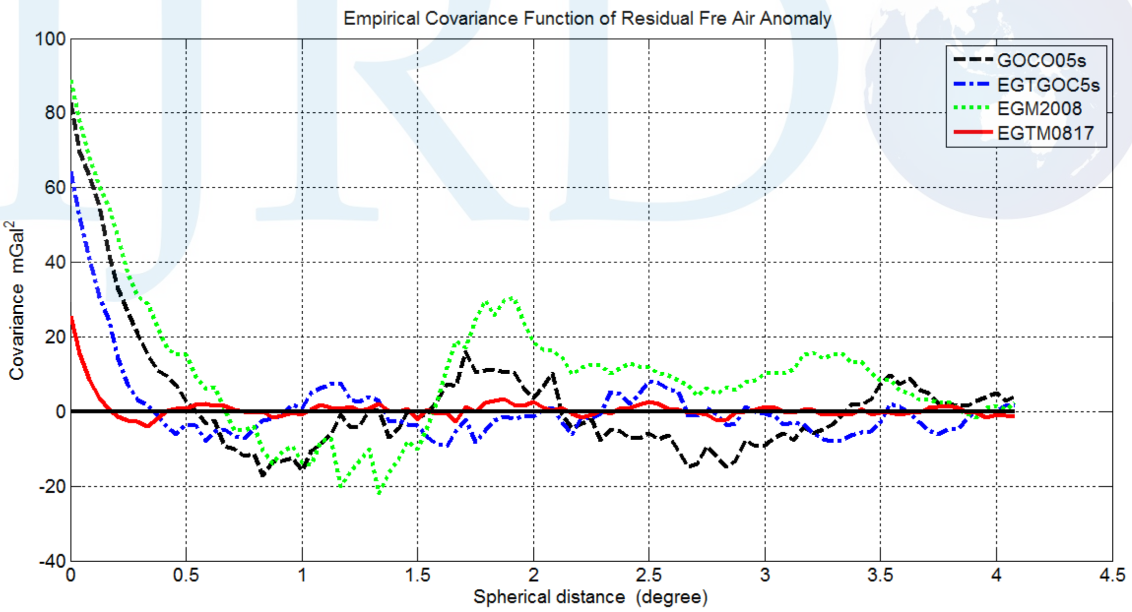

Figure 5: Empirical Covariance function for the residual free-air anomalies

From Fig. (5), both tailored model EGTGOC5s and EGTM0817 have a great smoothness effect on the residual free-air anomalies compared to their original model GOCO05s and EGM2008 respectively. Clearly, the smoothness of the residual is much obvious in the case of tailored model EGTM0817. Furthermore, the empirical covariance functions have three essential parameters (Amin, M. et al., 2002), which are the covariance or variance (C_0), the correlation length (α) and the first crossing-zero point ($\psi°_1$) see Table (6). In theory, the first crossing-zero point ($\psi°_1$) of the empirical covariance function of the reduced gravity anomalies up to degree N (Arabelos & Tscherning, 2010) should theoretically be located at distance equal 180º/ (2 × N), e.g. for degree 280, 560, and 2190, then ($\psi°_1$) should be located nearly at distances 19.3′, 9.7′ and 2.5′, respectively.

Table 6: Essential parameters of the empirical covariance function of the residual gravity anomalies

Models	Max. Degree.	Variance (C_0)	Correlation length (α)		First zero point ($\psi°_1$)	
		mgal^2	degrees	Km	degrees	minute
GOCO05s	280	82.602	0.17º	19	0.54º	32.4′
EGTGOC5s	280	64.188	0.12º	13	0.36º	21.6′
EGM2008	2190	88.776	0.22º	24	0.67º	40.2′
EGTM0817	2190	25.423	0.06º	6	0.18º	10.8′

Accordingly, from Fig.(5), and referring to Table (6), the correlation length and position of the first crossing-zero point for GOCO05s and EGM2008 models are much longer (in terms of spherical distance) than what expectedly justified by the degree and order of the expansion, this is due to the local test data and the models are not error-free. In addition, the satellite-only model GOCO05s does not contain any terrestrial gravity data; EGM2008 is also not complete to degree and order 2190. Besides, Egypt is located in the area, where collected data are either unavailable, or too sparse, and too inaccurate with low resolution (Pavlis et al., 2012), which have a negative impact during collected global gravity data for developing EGM2008. However, from Fig. (5), and Table (6), the actual appearance of the first crossing- zero point for EGTGOC5s and EGTM0817 tailored models are located at distances 21.6′ and 10.8′, which are really close to 19.3′ and 9.7′ that are theoretically justified for degrees 280 and 560, respectively. These improvements are due to a combination of the local gravity anomalies in these models. Finally from Table 6, if we compare the other values of the Empirical Covariance functions' parameters of both tailored models, we can see that the values of the variance (the covariance at zero spherical distance) and the correlation length are greatly improved for EGTM0817, where they were dropped by about 60% and 50%, respectively, comparing to those of EGTGOC5s.

6. Conclusions

- From the previous analysis of all results gained here in this study, we could conclude that the two tailored models EGTGOC5s and EGTM0817, which have been developed in this investigation, gives much better residual gravity anomalies than the original models. Consequently, this confirms that the computation of tailored models is a convenient and efficient way for taking new local gravity data into account when representing local fields if existing models do not appear to fit these data well.
- The comparison between both tailored models reveals a better accuracy for the EGTM0817 model, where the mean value, the standard deviation and the range of the reduced gravity anomalies to EGTM0817 have been decreased by about 80%, 30%, and 21%, respectively, compared with those of EGTGOC5s. Hence, the tailored model EGTM0817 could provide a better reference model for recovering the actual long-medium wavelength spectral information in Egyptian territory.
- Finally, we recommend that, using the EGTM0817 tailored geopotential model for Egypt to compute any of the various gravity field functions, such as geoidal heights, gravity anomalies, deflections of the vertical….etc.

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